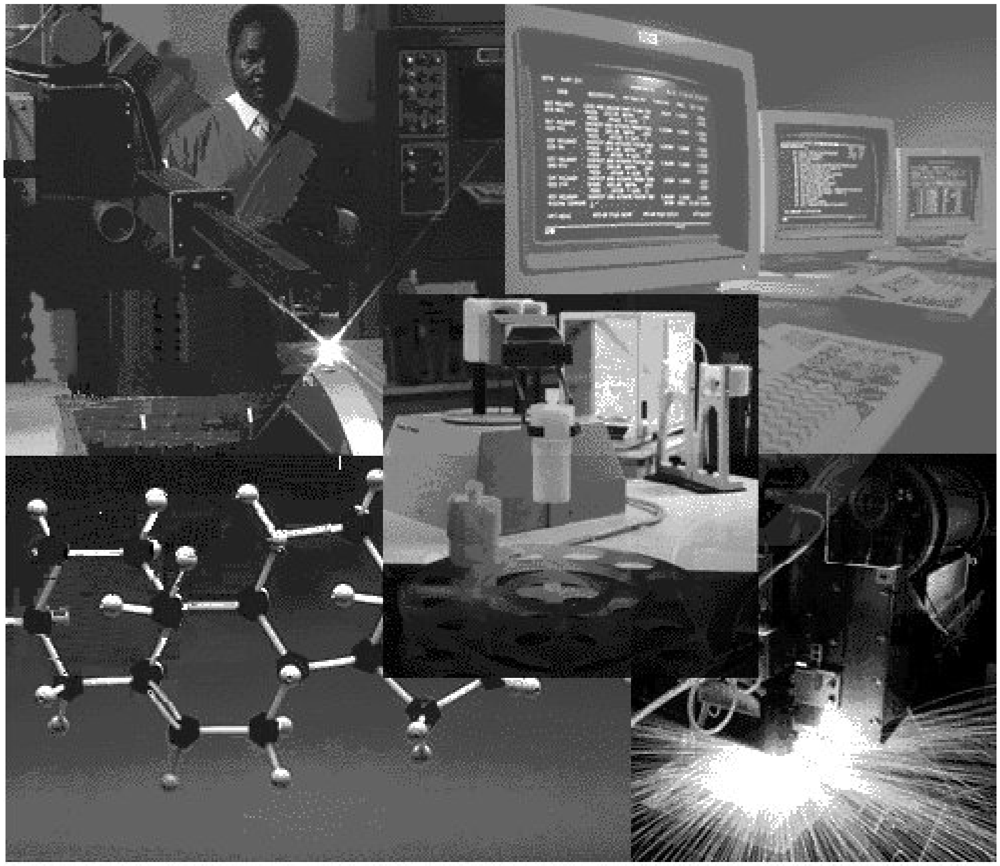
Vocational Math I
804-379



Text and Workbook

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How to use this workbook

Each of the six units consists of text plus worked examples. These are followed by **Exercises** . You should work through each unit checking the answers of the sample problems on your calculator. After this you should **work** the exercises. The answers are provided at the end of each unit. Avoid looking at the solution before attempting the problem yourself. This often leads to a false sense of understanding. A good practice might be to do all the problems, referring to the unit examples as necessary, and then checking the answers with the text. Any problems missed should then be reviewed closely until you fully understand the solution. **Good Luck!**

Calculator Use

Throughout most of human history computation has been a tedious task that was often postponed or avoided entirely. It is only in the last generation that the use of inexpensive handheld calculators has transformed the ways that people deal with quantitative data. Today the use and understanding of electronic computation is nearly indispensable for anyone engaged in technical work. There are a variety of inexpensive calculators available for student use. Some even have graphing and/or symbolic capabilities. Most newer model calculators such as the Casio models *fx-300W*, *fx-300MS*, *fx-115MS*, and the Texas Instruments models TI-30X IIB, TI-30X IIS, TI-34 II enter calculations in standard "algebraic" format. Older calculators such as the Casio *fx-250HC* and the Texas Instruments TI-30Xa and TI-36X enter some calculations in a "reverse" format. Both types of calculators are priced under twenty dollars, yet possess enough computational power to handle the problems faced in most everyday applications. The Casio *fx-300MS* is fairly representative of the "newer" format calculators and the TI-30Xa is typical of "older" format ones. While other calculator models have similar or even better features for performing the required computations, the reader will be responsible for learning their detailed use. **Never** throw away the user's manual!

In order to perform a computation the correct keystrokes must be entered. In order to indicate the sequence of keystrokes the following notation will be used. Digits (0 through 9 plus any decimal point) will be presented in normal typeface. Any additional keystrokes will be enclosed in boxes. For example, to multiply 7 times 8, the command sequence will be written as 7 [×] 8 [=] and 56 appears on the display.

There are eight rows (horizontal) below the gray keys and six columns (vertical) of **rectangular** keys on the Casio *fx-300MS* . The "times" key is in the sixth row, fourth column. The "equals" key is in the eighth row, fifth column, or the bottom right-most key. There are eight rows and five columns of keys on the TI-30Xa . The "times" key is in the fourth row, fifth column. The "equals" key is in the eighth row, fifth column, or again the bottom right-most key. The [=] key forces a computation and generally must be pressed at the end of any calculation.

The display on the Casio *fx-300MS* has two lines. The top line shows the key strokes entered, while the larger second line shows the result of the calculation. In the first row beneath the display are non-rectangular buttons. The REPLAY buttons allow you to recall and/or edit the most recently entered calculation.

Most keys on both calculators have a second function printed above each key. These secondary functions appear in gold on the Casio *fx-300MS* and in yellow on the TI-30Xa. To access these functions on the Casio *fx-300MS* first press the [SHIFT] key on the top left of the calculator. To access these functions on the TI-30Xa first press the [2nd] key also on the top left. To store a result on the Casio *fx-300MS* press the [STO] key (fourth row, first column) followed by one of six "registers" A through F accessed by selecting a key from the third row. To store a result on TI-30Xa press the [STO] key followed by one of the three registers 1 through 3 accessed by pressing the digit. To recall a stored result on either calculator, press the [RCL] button followed by the register key. On the Casio *fx-300MS* you can also access registers by pressing ALPHA from the row beneath the display followed by the register name (A – F or X, Y, M).

Mathematical expressions which involve more than one operation appear ambiguous. For example, is

$$5+6\times 2=11\times 2=22 \quad \textbf{or} \quad 5+6\times 2=5+12=17 \quad ?$$

To clarify this question, mathematics has developed the following hierarchy of computations called **order of operations**.

1. Perform all operations that appear in grouping symbols first. If grouping symbols are nested, do the innermost first.

2. Raise all bases to powers in the order encountered moving from left to right.

3. Perform all multiplications/divisions in the order encountered moving from left to right.

4. Perform all additions/subtractions in the order encountered moving from left to right.

Here grouping symbols means parentheses ( ), brackets [ ], braces { }, etc. An example of a nested expression is $(6+2\times(4+1))\div 8$. The innermost grouping symbol is (4+1) so the result is $(6+2\times 5)\div 8=(6+10)\div 8=16\div 8=2$. Raising a base to a power (also known as an **exponent)** means repeated multiplication as in

$$6^3=6\times 6\times 6=216.$$

To perform this calculation on the Casio *fx-300MS* enter the following keystrokes:

6 [^] 3 [=] . The [^] key is in the second row, fourth column of the Casio.

To perform this same calculation on the TI-30Xa enter the following keystrokes :

6 [y^x] 3 [=] . The [y^x] key is in the second row, fifth column of the TI-30Xa.

We will cover exponents in greater detail in Unit 3 **Decimal Fractions**.

In the original problem posed above the multiplication of 6 with 2 is performed before the addition of 5. The proper answer is therefore 17 . The other interpretation could be achieved by using parentheses $(5+6)\times 2 = 11\times 2 = 22$.

Order of operations is built into all scientific calculators. That is, if you enter the keystrokes in the correct order, the calculator will automatically perform the correct calculation. On both the Casio *fx-300MS* and the TI-30Xa entering

5 [+] 6 [×] 2 [=] yields the correct answer of 17 .

In many formulas *x* occurs as a variable, but then confusion with the times sign can result. To avoid this, alternative symbols for multiplication are used. They are the dot notation and adjacent parentheses as in $7\times 3 = 7\cdot 3 = (7)(3) = 21$. The Casio *fx-300MS* **recognizes** the adjacent parentheses as multiplication, but the TI-30Xa does **not**. On the TI-30Xa the times operation [×] must be inserted between the parentheses.

Division is also indicated by a variety of notations. For example, the following all mean 34 divided by 17,

$$34 \div 17 = 34/17 = \frac{34}{17} = 17\overline{)34} = 2 \quad .$$

In addition to parentheses, brackets and braces, certain symbols act as **implied grouping symbols**. The most important of these are the fraction bar and the square root symbol. The fraction bar acts to separate the numerator from the denominator. If either or both of the numerator or denominator consist of an expression with operations, these must be performed first before the division indicated by the fraction bar. For example,

$\frac{7+3}{2+3} = \frac{10}{5} = 2$. To perform this computation on the calculator, parentheses need to be inserted around both numerator and denominator as shown below.

[( ] 7 [+] 3 [ )] [÷] [( ] 2 [+] 3 [ )] [=]

The left parenthesis is in the fourth row, third column of the Casio *fx-300MS* and the right parenthesis is the next key over, in the fourth row, fourth column. On the TI-30Xa the left and right parentheses are in the fourth row, the third and fourth columns, respectively. Parentheses are the only grouping symbol the calculator recognizes or uses.

The square root symbol also acts as a grouping symbol. Any calculation inside the square root needs to be completed before the root is taken. For example,

$$\sqrt{25+144} = \sqrt{169} = 13 \ .$$

To perform this computation on the calculator parentheses need to be inserted around the expression inside the square root symbol.
On the Casio *fx-300MS* enter the following keystrokes:

[$\sqrt{\ }$] [( ] 25 [+] 144 [ )] [=] .

On the TI-30Xa enter the keystrokes:

[( ] 25 [+] 144 [ )] [$\sqrt{\ }$] [=] .

Note : On "newer" calculators like the Casio *fx-300MS* one enters expressions the way "they look" , i.e., the square root symbol comes first. On older models like the TI-30Xa some functions like [$\sqrt{\ }$] come after the expressions they are to evaluate. In any case, using parentheses keys when necessary is a good habit to acquire. Failure to do so usually results in wrong answers!

Exercises:

Perform the following arithmetical operations

1.	$47 + 271$	=________	2.	$19+67+43$	=________
3.	$269+151-237$	=________	4.	$135+932-612$	=________
5.	45×125	=________	6.	$167\times134-21679$	=________
7.	$72498\div129$	=________	8.	$68\times42\div136$	=________
9.	$12\times9-8\times7\div4\times2$	=________	10.	$12\times9-8\times7\div(4\times2)$	=________
11.	$4(19-12)\div2-11$	=________	12.	$32-3\times9+16\div4\times2$	=________
13.	$\frac{561}{51}$	=________	14.	$12+\frac{18}{3}-5$	=________
15.	$\frac{5+13+2}{1+3}$	=________	16.	$\frac{12\times5\times4}{6\times2+3}$	=________

From a board 10 feet long a piece 29 inches was cut off. How long is the piece remaining? Ignore the width of the cut.

17) ____________

In 1993 General Motors had total sales of $133,621,900,000 , while Ford Motor Company had sales of $108,521,000,000 . How much more were GM's sales than Ford's?

18) ____________

If 128 copies of a software package cost $5760, what is the cost per copy?

19) ____________

A carpenter earns $14 per hour plus time and a half for overtime (more than 40 hours in one week). If she works 58 hours one week, what is her gross pay for that week?

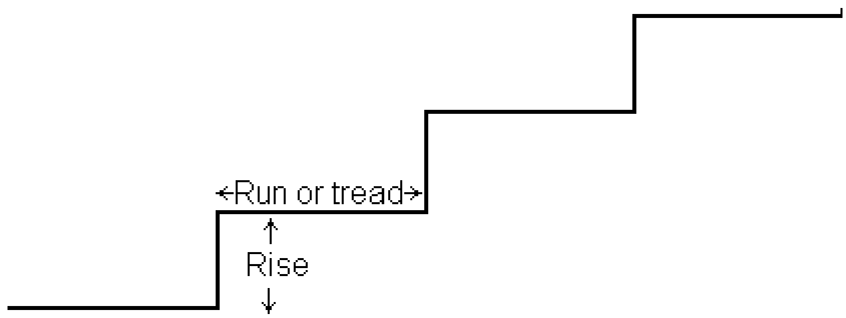
20) ____________

You order 15 CD's at $8 a CD and 24 cassette tapes at $3 a tape. What is the cost of the order?

21) ____________

A truck averages 16 miles per gallon and has a 25 gallon gas tank. What is the furthest distance the truck can travel without stopping for gas?

22) ____________



A stairway consists of rises of 6 in and must reach a height of 10 feet. How many rises are needed?

23) ____________

A stairway consists of 4 in rises and treads of 18 in. If the height of the stairs is 4 feet, what is the distance taken up by the stairway on the lower floor?

24) ____________

Answers:

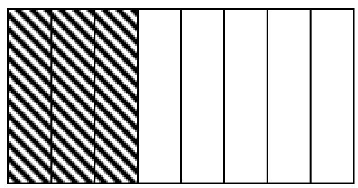
1. 318 ; 2. 129 ; 3. 183 ; 4. 455 ; 5. 5625 ; 6. 699 ; 7. 562 ; 8. 21 ; 9. 80 ; 10. 101
11. 3 ; 12. 13 ; 13. 11 ; 14. 13; 15. 5 ; 16. 16 ; 17. 91 inches ; 18. $25,100,900,000
19. 45 ; 20. $938 ; 21. $192 ; 22. 400 miles ; 23. 20 ; 24. 216 in or 18 ft

Fractions

Fractions are ratios of whole numbers which allow us to express numbers which are between the whole numbers. For example,

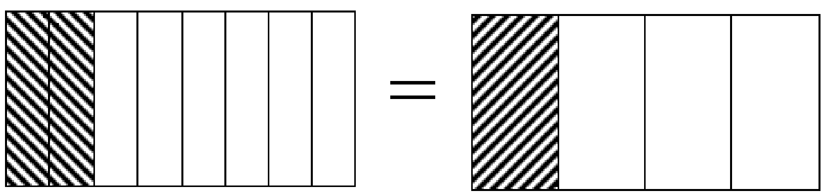
$$2\frac{2}{3} = 2 + \frac{2}{3} \text{ is between } 2 \text{ and } 3.$$

Fractions represent "part of a whole". Imagine that we have a freight car with eight equal sized compartments. If three of these compartments are full of grain, we would indicate that we have three eighths of a freight car's worth of grain. This is illustrated below.



$$= \frac{3}{8}$$

Consider a car with eight compartments of which two are full. The fraction of a full car is two eighths. If we look at the same car split into four equal compartments, this same amount of grain fills one fourth of the car. We arrive at the following result.



$$\frac{2}{8} = \frac{1}{4}$$

We say that such equal fractions while they "look different" are **equivalent**. To generate equivalent fractions, we can multiply or divide both numerator (the top number) and denominator (the bottom number) by a common number. So we have the following fractions equivalent to two thirds.

$$\frac{2}{3} = \frac{2\times 3}{3\times 3} = \frac{6}{9}$$
$$= \frac{2\times 17}{3\times 17} = \frac{34}{51}$$

Similarly, $\frac{18}{24} = \frac{18\div 6}{24\div 6} = \frac{3}{4}$. This same result could be stated in terms of "canceling" the common factor of 6 between the numerator and denominator. $\frac{18}{24} = \frac{6\times 3}{6\times 4} = \frac{\cancel{6}\times 3}{\cancel{6}\times 4} = \frac{3}{4}$.

If a fraction has no common factors between its numerator and denominator, the fraction is in lowest terms.

There are three types of fractions.

1. Proper fractions with the numerator less than (symbolized by <) the denominator. All proper fractions are less than 1 .

2. Improper fractions with the numerator greater than (symbolized by >) the denominator. All improper fractions are greater than 1 . Improper fractions can be expressed as a **mixed number**, which is a whole number plus a proper fraction. For example, $\frac{25}{6} = 6\overline{)25} = 4 + \frac{1}{6} = 4\frac{1}{6}$.

3. Unit fractions with the numerator equal to the denominator. All unit fractions are equal to one. For example,
$$\frac{19}{19} = \frac{4}{4} = \frac{25}{25} = \frac{1}{1} = 1 \ .$$

Note : when we write $4\frac{1}{6}$, we are using a shorthand notation. There really **is** a + sign between the 4 and the one sixth that's understood but unstated. Working backwards we can convert a mixed number into an improper fraction. For example,
$$7\frac{2}{3} = \frac{7 \times 3 + 2}{3} = \frac{23}{3} \ .$$

Fractions can be entered on both the Casio *fx-300MS* and the TI-30Xa using the [$a\frac{b}{c}$] key , which is found in the second row, second column of the Casio *fx-300MS* and the seventh row, first column of the TI-30Xa. For example, to enter the fraction $\frac{14}{24}$ use the following keystrokes: 14 [$a\frac{b}{c}$] 24 [=] .

[7⌋12] will then appear in the display as the fraction reduced to lowest terms.

For a mixed number such as $11\frac{5}{16}$, enter the following keystrokes :

11 [$a\frac{b}{c}$] 5 [$a\frac{b}{c}$] 16 [=] . The Casio *fx-300MS* displays [11⌋5⌋16] ,

while the TI-30Xa displays [11_5⌋16] . To change this answer to the improper

fraction $\frac{181}{16}$, enter [SHIFT] [$a\frac{b}{c}$] on the Casio *fx-300MS*, or [2nd] [$a\frac{b}{c}$] on the TI-30Xa. If a fraction has already been entered and appears on the display of the

Casio *fx-300W*, entering [$a\frac{b}{c}$] converts it to a decimal and if [$a\frac{b}{c}$] is pressed a second time the decimal is converted back to a fraction. Fraction – decimal conversions are performed on the TI-30Xa by entering [2nd] [←]. The [←] key is the back space key, which deletes characters in the display. It is found in the eighth row, first column of the TI-30Xa.

As an application, solve for the following missing numerator : $\frac{\quad}{12} = \frac{6}{8}$. As a first step reduce $\frac{6}{8}$ to lowest terms as $\frac{3}{4}$. So $\frac{\quad}{12} = \frac{3}{4}$. The first denominator 12 is three times the second denominator 4, so the missing numerator must be three times the second numerator 3 . The answer is that the missing numerator is 9.

To compare two fractions and determine which is larger, we can use the following procedure :

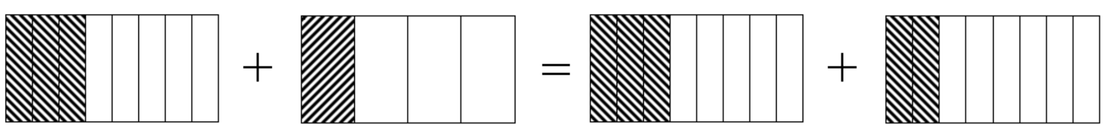
1. If the fractions involve mixed numbers with proper fractions, the number with the larger whole number is the larger number. For example,

   $7\frac{3}{16} > 5\frac{7}{8}$, since $7 > 5$.

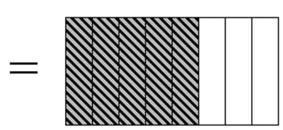
2. If the fractions are both proper fractions or mixed numbers with equal whole numbers, then convert the fractions into decimals. The number with the larger decimal is the larger number. For example,

   $5\frac{1}{3} < 5\frac{3}{8}$, since $\frac{3}{8} = 0.375 > 0.333... = \frac{1}{3}$.

To add or subtract fractions we need a common denominator. Consider adding $\frac{3}{8}$ to $\frac{1}{4}$. Since one fourth is equivalent to two eighths, we have the following solution :



$$\frac{3}{8} + \frac{1}{4} = \frac{3}{8} + \frac{2}{8}$$



$$= \frac{5}{8} .$$

If mixed numbers are involved, we first deal with the whole numbers, then the fractions. For example,

$$7\frac{1}{2}-5\frac{9}{16}=7-5+\frac{1}{2}-\frac{9}{16}=2+\frac{8}{16}-\frac{9}{16}=1+\frac{16}{16}+\frac{8}{16}-\frac{9}{16}=1+\frac{16+8-9}{16}=1\frac{15}{16}.$$

Note : Since $\frac{9}{16}>\frac{8}{16}$, we had to "borrow" $\frac{16}{16}$ from the 2.

To multiply fractions we form the product of the numerators over the product of the denominators. For example, $\frac{5}{8}\times\frac{3}{4}=\frac{5\times 3}{8\times 4}=\frac{15}{32}$.

If the product involves mixed numbers, we first convert them to improper fractions. For example,

$$2\frac{5}{6}\times 4\frac{1}{5}=\frac{2\times 6+5}{6}\times\frac{4\times 5+1}{5}=\frac{17}{6}\times\frac{21}{5}=\frac{17}{\cancel{3}\times 2}\times\frac{\cancel{3}\times 7}{5}=\frac{17\times 7}{2\times 5}=\frac{119}{10}=11\frac{9}{10}.$$

Note : we canceled the common factor of 3 between numerator and denominator in this calculation. In a multiplication problem this can always be done and saves the effort of later having to reduce the final answer. Also note that the answer is "reasonable" in that

$$2\frac{5}{6}\approx 3 \text{ and } 4\frac{1}{5}\approx 4 \text{ , so } 2\frac{5}{6}\times 4\frac{1}{5}\approx 3\times 4=12 .$$

A quick estimation like this can often catch silly mistakes even when using a calculator!

Consider the division problem $8\div 2=4$. This is the same as $\frac{8}{2}=\frac{8}{1}\times\frac{1}{2}=8\times\frac{1}{2}=4$.

More generally, any division problem can be expressed as

$$a\div b=\frac{a}{b}=\frac{a}{1}\times\frac{1}{b}=a\times\frac{1}{b} .$$

This means that division by the number b is equivalent to multiplication by the fraction $\frac{1}{b}$. The fraction $\frac{1}{b}$ is called the **reciprocal** of $b=\frac{b}{1}$. To form the reciprocal of a number we exchange the numerator with the denominator. In summary, division by a **non-zero** number equals multiplication by the reciprocal of that number.

In a division problem 0 is never allowed as the denominator or divisor. The reason for this is as follows.

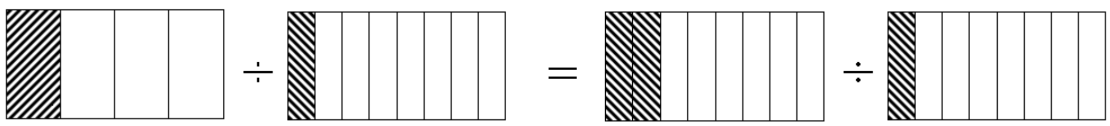
Suppose $20\div 0=\frac{20}{0}$ made sense. Then there would be some number, a , which is the answer to this division problem. Restating this as a multiplication problem would give

$$a\times 0=20 .$$

But any number times zero gives zero! So no sensible answer to $20\div 0=\frac{20}{0}$ exists.

Another way of explaining this goes to the very meaning of division. $20 \div 4 = \frac{20}{4} = 5$, means that 20 contains five 4's. How many 0's does 20 contain? There's no sensible answer to the question!

Consider now the division $\frac{1}{4} \div \frac{1}{8}$, from the diagram below it is clear that one fourth contains 2 one eighths. So the answer must be 2 .



$$\frac{1}{4} \div \frac{1}{8} = 2$$

$$\frac{1}{4} \times 8 = \frac{8}{4}$$

$$= 2$$

The following shows that this result is consistent with the multiplication by the reciprocal definition of division.

$$\frac{1}{4} \div \frac{1}{8} = \frac{1}{4} \times \frac{8}{1} = \frac{8}{4} = 2 \ .$$

If the division involves mixed numbers, we first convert them into improper fractions. For example,

$$4\frac{1}{2} \div 1\frac{1}{8} = \frac{9}{2} \div \frac{9}{8} = \frac{\cancel{9}}{2} \times \frac{8}{\cancel{9}} = \frac{8}{2} = 4 \ .$$

In expressions which combine operations the standard order of operations apply as shown in the following :

$$\frac{2}{3} \times 2\frac{1}{2} - 1\frac{1}{4} \div 3 = \frac{\cancel{2}}{3} \times \frac{5}{\cancel{2}} - \frac{5}{4} \times \frac{1}{3} = \frac{5}{3} - \frac{5}{12} = \frac{5 \times 4}{3 \times 4} - \frac{5}{12} = \frac{20-5}{12} = \frac{15}{12} = 1\frac{3}{12} = 1\frac{\cancel{3} \times 1}{\cancel{3} \times 4} = 1\frac{1}{4} \ .$$

These calculations are all easily performed on either the Casio *fx-300MS* or the TI-30Xa. The keystokes for the previous calculation are as follows :

2 [$a\frac{b}{c}$] 3 [$\times$] 2 [$a\frac{b}{c}$] 1 [$a\frac{b}{c}$] 2 [$-$] 1 [$a\frac{b}{c}$] 1 [$a\frac{b}{c}$] 4 [$\div$] 3 [$=$] .

More involved calculations with grouping symbols are also possible. For example,

$$3\frac{3}{4}-\left(5\frac{3}{16}-3\frac{7}{8}\right)\div 2\frac{1}{2}=3\frac{3}{4}-\left(2+\frac{3}{16}-\frac{14}{16}\right)\div\frac{5}{2}=3\frac{3}{4}-\left(1+\frac{16+3-14}{16}\right)\times\frac{2}{5}=$$

$$=3\frac{3}{4}-1\frac{5}{16}\times\frac{2}{5}=3\frac{3}{4}-\frac{21}{16}\times\frac{2}{5}=3\frac{3}{4}-\frac{21}{40}=3+\frac{30-21}{40}$$

$$=3\frac{9}{40}\quad .$$

This is keystroked as follows :

3 [$a\frac{b}{c}$] 3 [$a\frac{b}{c}$] 4 [−] [( ] 5 [$a\frac{b}{c}$] 3 [$a\frac{b}{c}$] 16 [−] 3 [$a\frac{b}{c}$] 7 [$a\frac{b}{c}$]
8 [ )] [÷] 2 [$a\frac{b}{c}$] 1 [$a\frac{b}{c}$] 2 [=] .

Exercises:

Write as an improper fraction.

$6\frac{11}{16}$ 1) ______________

$3\frac{7}{10}$ 2) ______________

Write as a mixed number reduced to lowest terms.

$\frac{19}{8}$ 3) ______________

$\frac{26}{10}$ 4) ______________

Reduce to lowest terms.

$\frac{9}{12}$ 5) ______________

$7\frac{18}{32}$ 6) ______________

Supply the missing numerators:

$\frac{3}{4} = \frac{?}{12}$ 7) ______________

$3\frac{3}{4} = \frac{?}{8}$ 8) ______________

Indicate which number is larger.

$\frac{19}{32}$ $\frac{5}{8}$ 9) ______________

$3\frac{7}{16}$ $2\frac{3}{4}$ 10) ______________

Perform the indicated operations and express the answer as a fraction in lowest terms:

$\frac{3}{16}\times\frac{4}{9}$ 11) ______________

$6\div\frac{2}{3}$ 12) ______________

$\frac{3}{8}+\frac{5}{8}$ 13) ______________

$\frac{13}{32}-\frac{9}{32}$ 14) ______________

$4\frac{1}{8}\times 2\frac{1}{4}$ 15) ______________

$\frac{5}{8}\div\frac{15}{16}$ 16) ______________

$5\frac{3}{8}+3\frac{11}{32}$ 17) ______________

$5\frac{3}{8} \div 2\frac{3}{4}$

18) ______________

$2\frac{1}{4} \times 3\frac{2}{3}$

19) ______________

$7\frac{19}{32} - 5\frac{3}{4} - 1\frac{5}{8}$

20) ______________

$3\frac{1}{2} \times 3\frac{1}{4} - 10\frac{5}{8}$

21) ______________

$1\frac{3}{8} - 4\frac{3}{4} \div 4$

22) ______________

Solve and state all results as fractions reduced to lowest terms.

How many pieces of $\frac{5}{16}$ inch thick plywood are in a stack 35 inches high?

23) ______________

A lumberyard sells lumber only in even foot lengths. What is the shortest single board of lumber from which a carpenter could cut three $3\frac{1}{4}$ ft long and two $2\frac{3}{4}$ ft long pieces?

24) ______________

A cubic foot contains about $7\frac{1}{2}$ gallons. How many cubic feet are there in 120 gallons?

25) ______________

A nail $3\frac{1}{2}$ inches long, goes through a board $2\frac{3}{8}$ inches thick supporting a joist. How far into the joist does the nail extend?

26) ______________

A part is measured as $2\frac{7}{16}$ inches on a scale drawing. If the scale is one ft to $\frac{1}{2}$ inch, how long is the actual part?

27) ______________

Answers :

1. $\frac{107}{16}$; 2. $\frac{37}{10}$; 3. $2\frac{3}{8}$; 4. $2\frac{3}{5}$; 5. $\frac{3}{4}$; 6. $7\frac{9}{16}$; 7. $\frac{9}{12}$ missing numerator is 9
8. $\frac{30}{8}$ missing numerator is 30 ; 9. $\frac{5}{8} > \frac{19}{32}$; 10. $3\frac{7}{16} > 2\frac{3}{4}$; 11. $\frac{1}{12}$; 12. 9
13. 1 ; 14. $\frac{1}{8}$; 15. $9\frac{9}{32}$; 16. $\frac{2}{3}$; 17. $8\frac{23}{32}$; 18. $1\frac{21}{22}$; 19. $8\frac{1}{4}$; 20. $\frac{7}{32}$
21. $\frac{3}{4}$; 22. $\frac{3}{16}$; 23. 112 pieces ; 24. A board 16 ft long ; 25. 16 cubic feet
26. $1\frac{1}{8}$ in 27. $4\frac{7}{8}$ ft

Decimal Fractions

The difficulty in adding or subtracting fractions "by hand" compared to adding or subtracting whole numbers is obvious to anyone who has done such calculations. This difficulty motivated the development of representing fractions as decimal numbers. Using decimal fractions all arithmetical operations are similar to computations with whole numbers. The only complication is keeping track of the position of the decimal point.

The basis of the decimal representation of numbers is the use of **place value**. This allows us to represent an infinite range of numbers with only ten symbols (the digits 0 through 9). Contrast this with Roman Numerals or other early number systems where new symbols are constantly added to represent larger values. Place value uses the powers of 10.

10^0	=	1	(The **definition** of an exponent of 0)
10^1	=	10	
10^2	=	100	
10^3	=	1,000	
10^4	=	10,000	
10^5	=	100,000	
10^6	=	1,000,000	
etc.			

Note : 10^n is equal to 1 followed by n zeros.

When we write a number such as 27,483, the digit 2 stands not for 2, but for $2(10^4) = 20,000$. The digit 7 represents $7(10^3) = 7000$, etc. The value we associate with each digit comes from its place in the number. The right most digit of a whole number is in the "one's place", the second digit from the right is the "ten's place", etc. To extend the decimal system to fractions, we use the reciprocal powers of 10 and the decimal point to separate the "one's place" from the "tenth's place".

$$10^{-1} = \frac{1}{10^1} = 0.1$$

$$10^{-2} = \frac{1}{10^2} = 0.01$$

$$10^{-3} = \frac{1}{10^3} = 0.001$$

$$10^{-4} = \frac{1}{10^4} = 0.0001$$

$$10^{-5} = \frac{1}{10^5} = 0.00001$$

$$10^{-6} = \frac{1}{10^6} = 0.000001$$

In general $10^{-n} = \frac{1}{10^n}$ is a decimal point followed by $n-1$ zeros.

The leading 0 to right of the decimal point is not required for a number smaller than 1. It is used to emphasize the location of the decimal point. A decimal fraction such as 0.375 is interpreted as

$$0.375 = 3\times10^{-1} + 7\times10^{-2} + 5\times10^{-3}$$
$$= \frac{3}{10} + \frac{7}{100} + \frac{5}{1000} = \frac{375}{1000} = \frac{3\times125}{8\times125} = \frac{3}{8} \quad .$$

Note : adding extra zeros to the right of the rightmost digit to the right of the decimal point does **not** change the value of the decimal fraction. It does, however, imply a greater knowledge of the precision of the value.

A decimal fraction like 0.375 is called a **terminating** decimal because the digits to the right of the decimal point come to an end. The procedure outlined above is how to convert a terminating decimal to a fraction. It is summarized below:

1. Carry along the digits to the left of the decimal point as the whole number part of the resulting mixed number. If there are no non-zero digits to the left of the decimal point, the decimal represents a proper fraction.

2. Put the digits to the right of the decimal point over the power of 10 that goes with the right most decimal place. For example, in converting 0.1145, 1145 is put over 10,000 since the right most digit, 5, is in the ten-thousandth's place.

$$0.1145 = \frac{1145}{10000} = \frac{\cancel{5}\times229}{\cancel{5}\times2000} = \frac{229}{2000} \quad .$$

3. Reduce this fraction to lowest terms.

To convert a fraction to a decimal is quite easy. We just translate the fraction bar into a division. Remember that in a mixed number there is an understood but unstated plus sign. So that

$$7\frac{11}{16} = 7 + 11\div16 = 7.6875 \quad .$$

This can also be done directly using the Casio *fx-300MS* or TI-30Xa as was discussed in Unit 2.

If a fraction is in lowest terms and its denominator has a factor besides 2 or 5, then that fraction, when converted to a decimal, will generate a **repeating** decimal. For example,

$$\frac{5}{12} = \frac{5}{2\times2\times3} \text{ , so 12 has a factor of 3 and } 5\div12 = 0.416666... = 0.41\overline{6} \quad .$$

The 6's as indicated either by the ellipsis "…" or 6 with a bar on top repeat "forever".

$$\frac{5}{12} = 0.416666... = 0.41\overline{6} = 0.416\overline{6} = 0.41666\overline{6} .$$

Note: all of these ways of writing the repeating decimal are the same. Calculators will display 0.416666667 since they work with a fixed number of digits and will round the last digit displayed.

To convert a repeating decimal into a fraction is a little complicated and is rarely encountered in practical problems. As a result no problems requiring such a conversion occur in the unit exercises. However, if you are **curious**, the procedure is summarized and illustrated below :

1. Count and record the number of decimal places from the decimal point to the repeating string of digits.

2. Move the decimal point to the left by this number of places. The result is a decimal number where the repeating pattern of digits begins in the tenth's place immediately to the right of the decimal point.

3. The digits to the left of the decimal point of the result from Step 2 become the whole number part of a mixed number. If there are no non-zero digits to the left of the decimal point, then the original decimal began the repeating pattern with the first digit and the whole number part of the mixed number is zero.

4. Add the whole number from Step 3 to a fraction with the repeating digits as the numerator and a string of 9's as the denominator. The number of 9's in the string is equal to the number of repeating digits in the numerator.

5. Take the fraction from Step 4 and divide it by 10 raised to the power of the number from Step 1. This number, worked out as a fraction, is the fraction equivalent to the original repeating decimal.

To illustrate the steps convert 0.00666… to a fraction.
Step 1. The number of places from the decimal point to the repeating string of 6's is two.
Step 2. The result is the decimal 0.666… .
Step 3. The whole number is 0 .
Step 4. There is one repeating digit, a 6 , so the result is $0+\frac{6}{9}=\frac{2}{3}$.
Step 5. Dividing two thirds by $10^2 = 100$ gives

$$\frac{2}{3}\div 10^2 = \frac{2}{3}\div 100 = \frac{2}{3}\times\frac{1}{100} = \frac{\cancel{2}\times 1}{\cancel{2}\times 50\times 3} = \frac{1}{150} \quad \text{. So } 0.00\overline{6} = \frac{1}{150} \quad .$$

As a more complicated example consider converting 3.1527272727… to a fraction.
Step 1. The number of places from the decimal point to the repeating string of 27's is two.
Step 2. The result is the decimal 315.272727… .
Step 3. The whole number is 315 .
Step 4. There are two repeating digits, 27 , so the result is $315+\frac{27}{99}=315+\frac{\cancel{9}\times 3}{\cancel{9}\times 11}=315\frac{3}{11}$.
Step 5. Dividing the answer of Step 4 by $10^2 = 100$ gives

$$315\frac{3}{11}\div 100 = \frac{3468}{11}\times\frac{1}{100} = \frac{\cancel{4}\times 867}{11}\times\frac{1}{\cancel{4}\times 25} = \frac{867}{275} = 3\frac{42}{275} \quad .$$

Using a calculator we can verify that $3\frac{42}{275} = 3 + 42 \div 275 = 3.15272727... = 3.17\overline{2}\overline{7}$.

Often we wish to approximate a decimal number by finding another decimal roughly equal to the first number, but expressed with less digits. This process is called **rounding**. To round use the following procedure :

1. Determine the decimal place to which the number is to be rounded. Often this is stated in the problem or application.

2. If the digit to the right of this decimal place is less than 5, then replace all digits to the right of this decimal place by zeros or discard them if they are to the right of the decimal point.

3. If the digit to the right of the decimal place is 5 or greater, then increase the digit in this decimal place by 1 and replace all digits to the right of this decimal place by zeros or discard them if they are to the right of the decimal point.

As an example, consider rounding 10,547.395 to the different decimal places shown in the following table.

10,547.395 rounded to	**Decimal Place of Rounding**	**Result**
2 places	hundredth's place	10,547.40
1 place	tenth's place	10,547.4
the nearest unit	one's place	10,547.
the nearest ten	ten's place	10,550
the nearest hundred	hundred's place	10,500
the nearest thousand	thousand's place	11,000

Raising numbers to powers or exponents occurs in many applications. Recall from Unit 1 that b^n means a product of n factors of b . The number b is called the **base**, and n is the **power** or **exponent.**

$$\text{So } 1.574^5 = 1.574 \times 1.574 \times 1.574 \times 1.574 \times 1.574 = 9.661034658 \ .$$

This result is correct to as many places as the Casio *fx-300MS* or TI-30Xa display. To perform this calculation on the Casio *fx-300MS* use the keystrokes

1.574 [^] 5 [=], while on the TI-30Xa enter 1.574 [y^x] 5 [=] .

Newer and/or graphing calculators generaly use the "carrot" symbol ^ for exponents.

Exponents of two and three are very common and have special names; b^2 is called "b **squared**"

and b^3 is called "b **cubed**". Both the Casio *fx-300MS* and the TI-30Xa have [x^2] keys to square a number. On the Casio *fx-300MS* it is found in the second row, third column, while on the

TI-30Xa it is in the third row, third column. When evaluating an expression, the standard order of operations requires that bases be raised to powers before any multiplications or divisions are performed. This hierarchy is built into scientific calculators. For example, consider evaluating

$$3.54\times 7.21^3 - (10.7\times 6.28)^2 \div 3.56 \quad .$$

On the Casio *fx-300MS* this is done with the following keystrokes :

3.54 [×] 7.21 [^] 3 [−] [( ] 10.7 [×] 6.28 [ )] [x^2] [÷] 3.56 [=] .

The display shows the answer as 58.46760266 . The keystrokes on the TI-30Xa are identical except that the [y^x] key is used instead of the [^] key. The Casio *fx-300MS* does have an [x^3] key in the first row, sixth column, and this key could have been used instead of [^] 3 above.

Consider evaluating 25^{12} . Entering 25 [x^y] 12 [=] on the Casio *fx-300MS* gives the display 5.960464477x10^{16}. Entering 25 [y^x] 12 [=] on the TI-30Xa results in 5.960464478 ^{16} .

Because of the large size of the number both calculators have expressed the result in **scientific notation**. In scientific notation we express the answer as a decimal number between 1 and 10 times ten to a power. Here the number between 1 and 10 is 5.960464478 and the power on 10 is 16. In ordinary decimal notation, which the calculator can't display for lack of space, this answer would be written as 59,604,644,780,000,000 . If you try to work with these large decimal numbers, the advantages of scientific notation soon become obvious! **Note:** some calculators seem to suggest that the exponent applies to 5.960464478 . This is **not true**. The exponent is on ten, but to save space in the display the calculator does not show the 10.

Now consider $(0.04)^{12}$. The Casio *fx-300MS* displays 1.6777216x10^{-17}, while on the TI-30Xa it is 1.6777216 $^{-17}$. The result is in scientific notation with a negative exponent on 10. In ordinary decimal notation this result would be 0.000000000000000016777216 .The left-most non-zero digit, 1, is 16 (17–1) decimal places to the right of the decimal point. Thus, in scientific notation a positive exponent on 10 gives the number of decimal places the decimal point must move to the right to get the ordinary decimal answer, while a negative exponent on 10 gives the number of decimal places the decimal point must move to the left to get the ordinary decimal answer.

To enter a number in scientific notation on the Casio *fx-300W*, use the [EXP] key found in the bottom row, third column. For example, to enter 6.02×10^{23} , use the following keystrokes:

6.02 [EXP] 23 . A very small number like 7.15×10^{-12} is entered with

7.15 [EXP] [(–)] 12 . Here [(–)] is the "change sign" or minus key found in the third row, first column.

The procedure used on the TI-30Xa is identical except that the [EE] key is used instead of the [EXP] key and the change sign key is [+ ⇄ −]. The [EE] key is in the fourth row, second column and the change sign key is in the bottom row, fourth column.

Consider a table of squares of the whole numbers.

N	**N^2**
0	0
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64
9	81
10	100
11	121
12	144

If we reverse this table, i.e., start with N^2 and get the value of N, the table would look like.

N^2	**N**
0	0
1	1
2	1.41421356
3	1.732050808
4	2
5	2.236067977
6	2.449489743
7	2.645751311
8	2.828427125
9	3
10	3.16227766
11	3.31662479
12	3.464101615

The second number is called the square root of the first. In symbols

$$N = \sqrt{N^2} \text{ , for example, } 3 = \sqrt{9} \text{ .}$$

Remember from Unit 1 that the square root symbol acts as a grouping symbol. Any operations inside the square root need to be completed before the root is taken. For example,

$$\sqrt{116-16}=\sqrt{100}=10\ .$$

To perform this computation on the calculator parentheses need to be inserted around the expression inside the square root symbol. On the Casio *fx-300MS*enter

[√] [( ] 116 [−] 16 [ )] [=] , while [( ] 116 [−] 16 [ )] [√] [=]

are the corresponding keystrokes on the TI-30Xa.

A similar table of the cubes of whole numbers can be formed.

N	**N^3**
0	0
1	1
2	8
3	27
4	64
5	125

If we reverse this table, i.e., start with N^3 and get the value of N, the table would look like.

N^3	**N**
0	0
1	1
2	1.25992105
3	1.44224957
4	1.587401052
5	1.709975947
6	1.817120593
7	1.912931183
8	2

The second number is called the cube root of the first. In symbols

$$N=\sqrt[3]{N^3}\ \text{, for example, }\ 3=\sqrt[3]{27}\ .$$

On the Casio *fx-300MS* the cube root key is in first row, fourth column (use the “shift” key). On the TI-30Xa enter [2nd] 0 .

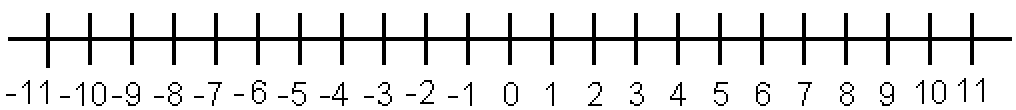
The cube root, like the square root, acts as a grouping symbol. Any operations inside the cube root need to be completed before the root is taken. For example,

$$\sqrt[3]{85\times 2-45}=\sqrt[3]{170-45}=\sqrt[3]{125}=5\ .$$

To perform this computation on the calculator parentheses need to be inserted around the expression inside the cube root symbol. On the Casio *fx-300MS* enter

[$\sqrt[3]{\ }$] [( ] 85 [×] 2 [−] 45 [ )] [=] , while the keystrokes on the TI30-Xa are [( ] 85 [×] 2 [−] 45 [ )] [2nd] 0 [=] .

When we are using numbers to express the change in a quantity, such as the amount of money in a checking account or a running back's total yards, we soon find that the quantities under study don't always increase. Bank accounts sometimes decline and running backs can lose yards! To represent a change, which decreases, we use negative numbers, while positive numbers represent an increase. A convenient way to visualize positive and negative numbers is the number line shown below. Here the positive (or "ordinary") numbers are to the right of zero and the negative numbers are to the left of zero.



The opposite of 5 is –5 since –5 + 5 = 0 . For example, if you lose $5 then make $5, you're back to zero. By the same argument the opposite of –5 is 5 . If a running back gains 10 yards, then loses 7, his net yardage is 3 . In symbols, 10 + (–7) = 3 . So adding a negative 7 is the same as subtracting a positive 7 . Also 10 + (–7) = (–7) + 10 = 3 . In general, a + (–b) = a – b, i.e., subtracting is the same as adding the opposite and visa versa .

Suppose a running back loses 3 yards every time he carried the ball. If he had four carries, his net yardage is –3 + (–3) + (–3) + (–3) = –12 . [You may have noticed that we don't write + –3 , but rather + (–3) , this is just to avoid the potential confusion of two adjacent operation symbols.] However, using the definition of whole number multiplication as repeated addition, we see that 4(–3) = –3 + (–3) + (–3) + (–3) = –12 . So a positive number times a negative number should result in a negative number. What about a negative times a negative? One of the fundamental rules of arithmetic is called the distributive property. It says that

$$a\cdot(b+c)=a\cdot b+a\cdot c \quad .$$

For example,

$$5\cdot(4+7)=5\cdot 4+5\cdot 7=20+35$$
$$5\cdot 11\ =\ 55.$$

Now,
$$(-1)\cdot(1+(-1)) = (-1)\cdot 1 + (-1)\cdot(-1)$$
$$(-1)\cdot(0) = -1 + (-1)\cdot(-1)$$
$$0 = -1 + (-1)\cdot(-1) \quad .$$

So $(-1)(-1)$ added to -1 gives zero. But only 1 added to -1 makes zero. So we conclude that

$$\mathbf{(-1)(-1) = 1} \quad .$$

In general, a negative number times a negative number gives a positive number. We have analogous statements in English. If I say "I am not dishonest", the double negative makes the sentence equivalent to saying "I am honest".

We now have another way of forming the opposite of any number, simply multiply by -1 , i.e., $-b = (-1)b$. The standard order of operations requires that we square before multiplication. This means that $-5^2 = -1(5^2) = -25$, while $(-5)^2 = (-5)(-5) = 25$. Consider now subtracting a negative number as in $10 - (-8) = 10 + (-1)(-8) = 10 + 8 = 18$. So subtracting a negative is the same as adding the positive.

Finally, division is the opposite operation to multiplication. Since $(-5)(6) = -30$ and $(-5)(-6) = 30$, then
$$(-30) \div (-5) = 6$$
$$(-30) \div 6 = -5$$
$$30 \div (-5) = -6$$
$$30 \div (-6) = -5 \quad .$$

In general, a negative number divided by a negative number is a positive number, while a negative divided by a positive or a positive divided by a negative is negative.

To enter a negative number on the Casio *fx-300MS* use the minus sign key [(−)] **before** the number just as it is written. For example, to evaluate

$$\frac{6\times(-10)}{-3-2} = \frac{-60}{-5} = 12$$

enter the following keystrokes:

[( ] 6 [×] [(−)] 10 [ )] [÷] [( ] [(−)] 3 [−] 2 [ )] [=] .

The keystrokes for the TI-30Xa are shown below. **Note:** you enter the negative numbers "backwards", i.e., first enter the value, then the change sign key.

[( ] 6 [×] 10 [+⇄−] [ )] [÷] [( ] 3 [+⇄−] [−] 2 [ )] [=] .

Exercises:

Perform the indicated operations giving answers to the stated number of decimal places:

$7.1164 + 3.3489$ (four places) 1) ______________

$27.32 - 6.972$ (two places) 2) ______________

0.25×0.4333 (two places) 3) ______________

$7.123 \div 1.48$ (three places) 4) ______________

$\$57.23 - \$7.89$ (two places, i.e., nearest penny) 5) ______________

1.79^2 (two places) 6) ______________

$\sqrt{0.144}$ (three places) 7) ______________

$2.53^2 \times 1.96 - 5.36^2 \div 2.89$ (two places) 8) ______________

$\sqrt{17^2 - 13^2}$ (one place) 9) ______________

$\sqrt[3]{7 \times 8 - 11}$ (two places) 10) ______________

$11.17^2 \times 5.10 - 5.97^3 \div 2.17$ (one place) 11) ______________

Write the following fractions as decimals:

$2\frac{11}{16}$ 12) ______________

$1\frac{7}{12}$ 13) ______________

Write the following decimals as a fraction in lowest terms:

0.45 14) ______________

8.84 15) ______________

Calculate the following:

$-56 \div 8$ 16) ______________

$(-8)^2 \div (-16)$ 17) ______________

$-8^2 \div (-16)$ 18) ______________

$\frac{(-5)(-4)}{-10}$ 19) ______________

$\frac{(-2)^3(-3)}{-6}$ 20) ______________

Solve the following problems :

A stack of eighteen pieces of lumber is 31.50 inches thick. How thick would a stack of thirty-three such sheets be?

21) ______________

A delivery truck gets 11.3 miles per gallon of gasoline. If gas costs $1.12 per gallon, what will be the cost of the gasoline needed to drive 189 miles?(Round to the nearest penny.)

22) ____________

A machinist earns $12.50 an hour plus time and a half for overtime (hours worked beyond 40). What is the machinist's gross pay for a 53.75 hour work week? (Round to the nearest penny.)

23) ____________

In the first six months of the year, Precision Auto Body had the following profit and loss record:

January	$8,736.52	profit
February	$12,567.34	profit
March	$1,282.72	loss
April	$478.68	profit
May	$179.66	loss
June	$1,257.23	profit

Find the total profit or loss for this six month period.

24) ____________

A welder earned $468.75 (gross pay before deductions) for 37.5 hours of work. Find her hourly rate of pay.

25) ____________

A cubic foot holds 7.481 gallons. A car has a gas tank which holds 14.5 gallons. To three decimal places, how many cubic feet is this?

26) ____________

Answers :

1. 10.4653 ; 2. 20.35 ; 3. 0.11 ; 4. 4.813 ; 5. $49.34 ; 6. 3.20 ; 7. 0.379 ; 8. 2.60 ; 9. 11.0

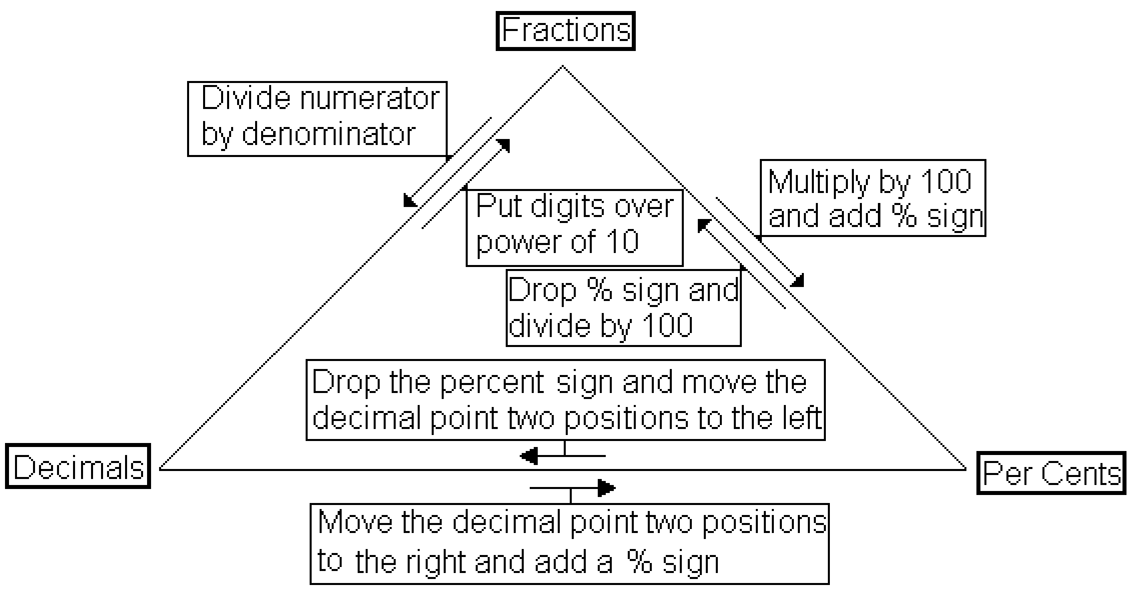
10. 3.56 ; 11. 538.3 ; 12. 2.6875 ; 13. 1.58333… ; 14. $\frac{9}{20}$; 15. $8\frac{21}{25}$; 16. –7

17. –4 ; 18. 4 ; 19. –2 ; 20. –4 ; 21. 57.75 in = 4 ft $9\frac{3}{4}$ in ; 22. $18.73 ; 23. $757.81

24. a profit of $21,577.39 ; 25. $12.50 ; 26. 1.938 cubic feet

Percent Problems

The word percent comes from Latin and literally means divide (per) by 100 (cent). Even the % symbol, if you look closely at it, resembles /100 . Thus, percents are just another way of writing fractions or decimal fractions. In the chart below are shown the conversions between fractions, decimals, and percents. A more detailed discussion for converting fractions into decimals and decimals into fractions was provided in Unit 3.



As examples consider the following :

$$28\% = 28 \div 100 = 0.28 \quad \text{(decimal answer)}$$

$$= \frac{28}{100} = \frac{7}{25} \quad \text{(fraction answer)}$$

$$0.145 = 14.5\% = 14\frac{1}{2}\% \quad \text{(percent answer)}$$

$$= \frac{145}{1000} = \frac{29 \times \cancel{5}}{200 \times \cancel{5}} = \frac{29}{200} \quad \text{(fraction answer)}$$

$$\frac{1}{6} = 1 \div 6 = 0.166\overline{6} \quad \text{(decimal answer)}$$

$$= \frac{1}{6} \times 100\% = \frac{50}{3}\% = 16\frac{2}{3}\% = 16.667\% \quad \text{(percent answer)}$$

$$\frac{1}{4}\% = 0.25\% = 0.25 \div 100 = 0.0025 \quad \text{(decimal answer)}$$

$$= \frac{1}{4} \div 100 = \frac{1}{4} \times \frac{1}{100} = \frac{1}{400} \quad \text{(fraction answer)} .$$

As an application of these methods consider the following problem.

The reduction percentage for automotive paint is the amount (volume) of thinner added divided by the amount (volume) of paint originally present. For a 150% reduction how many gallons of thinner must be added to 7.5 gallons of paint?

The solution is that

$$150\% = \frac{\text{volume thinner}}{\text{volume paint}} = \frac{\text{volume thinner}}{7.5\,\text{gallons}} = \text{volume thinner} \times \frac{1}{7.5\,\text{gallons}}$$

$$\text{volume thinner} = 150\% \div \frac{1}{7.5\,\text{gallons}} = 1.50 \times 7.5\,\text{gallons} = 11.25\,\text{gallons} = 11\frac{1}{4}\,\text{gallons} \ .$$

There are three basic percent problems.

1. Missing part.

2. Missing total amount.

3. Missing percent.

All three types of problems can be solved by the same strategy of translating word problems into math and a “little bit” of algebra. The translation is as follows :

what	means	$\square$	(an unknown, often called x)
is	means	$=$	
of	means	$\times$	(multiplication)

The “little bit” of algebra is as follows. Suppose you wanted to know $\square$ from the fact that

$$4 \times \square = \square \times 4 = 28 \ .$$

The “obvious” answer is 7 . How did we arrive at this? The opposite of multiplication is division. So

$$\square = 28 \div 4 = 7 \ .$$

This solution can be used as a template to solve any similar problem.

If $a \times \square = \square \times a = b$. Then $\square = b \div a = \frac{b}{a}$.

The following six problems illustrate the method.

1. What is 16% of \$140 ? (A missing part problem.)

Solution: $\square = 16\% \times \$140 = 0.16 \times \$140 = \$22.40$.

2. 27 is 54% of what value? (A missing total amount.)

Solution: $27 = 54\% \times \square = 0.54 \times \square \Rightarrow \square = 27 \div .54 = 50$.

3. What percent of \$160 is \$200? (A missing percent.)

Solution: $\square \times \$160 = \$200 \Rightarrow \square = \$200 \div \$160 = 1.25 = 125\,\%$.

4. What percent of $\frac{1}{4}$ is $\frac{3}{32}$? (A missing percent.)

Solution: $\square \times \frac{1}{4} = \frac{3}{32} \Rightarrow \square = \frac{3}{32} \div \frac{1}{4} = \frac{3}{32} \times \frac{4}{1} = \frac{3}{8} = \frac{3}{8} \times 100\,\% = 37\frac{1}{2}\,\%$.

5. 125% of what value is \$423 ? (A missing total amount.)

Solution: $125\% \times \square = \$423 \Rightarrow \square = \$423 \div 1.25 = \$338.40$.

6. 19 is to 80 like what number is to 300?

Solution: This problem is not strictly a percent problem. However, its solution is very similar in setup to a percent problem.
Here the phrase 19 "is to" translates into the fraction $\frac{19}{80}$, and the phrase "like" translates as = , so this statement becomes

$$\frac{19}{80} = \frac{\square}{300} = \square \times \frac{1}{300} \Rightarrow \square = \frac{19}{80} \div \frac{1}{300} = \frac{19}{80} \times \frac{300}{1} = 71\frac{1}{4} = 71.25 .$$

There are many common percent applications. In most communities consumers pay sales tax on certain purchases. If this tax is 5.5%, then the amount of sales tax is (0.055)(sales price). The checkout price is then the sales price + 5.5% of sales price or 105.5% of sales price. For example, suppose we purchase an item listed at \$89.99 .

The sales price	=	\$89.99	=	\$89.99	=	\$89.99
The sales tax	=	(0.055)(\$89.99)	=	\$4.94945	=	\$ 4.95
				Checkout	=	\$94.94

A faster way to compute the checkout price is to compute 105.5% of \$89.99 = (1.055)(\$89.99) = \$94.93945 which, rounded up to the next penny, is \$94.94 .

Consider the problem below.

The checkout price of a new TV was $258.48 . If 5.5% sales tax was charged, what was the sales price? The solution is as follows :

$$\text{Purchase Price} = 105.5\% \times \text{Sales Price} = 1.055 \times \text{Sales Price} \quad \Rightarrow \text{Sales Price} = \text{Purchase Price} \div 1.055$$

$$\text{Sales Price} = \$258.48 \div 1.055 = \$245.00 \; .$$

Occasionally retailers offer discounts on items. A 10% discount means that 10% of the list price has been removed in setting the sales price. If there had been **no** discount, the sales price would be 100% of the list price. The discount of 10% means that the new sales price is 100% – 10% = 90% of list price. This illustrates an important principle, when considering a percent change, the "base line" is 100% . Thus, a 12% increase means that the new value is 112% of the old value. Working backwards we get the following formula :

$$\text{Percent change} = \frac{\text{new value} - \text{old value}}{\text{old value}} \times 100\% \; .$$

Note: multiplying by 100% converts the fractional change into a percent change. If the percent change is positive, this is called a percent increase, if negative, a percent decrease. Consider the following examples.

A VCR regularly priced at $199.99 is discounted 20% . What is the new selling price?
Solution: $\text{New Sales Price} = (100\% - 20\%) \times \$199.99 = 0.80 \times \$199.99 = \$160.00 \; .$

If 5.5% sales tax is included, what is the checkout price of a VCR discounted 20% from its regular price of $199.99?
Solution: $\text{Checkout Price} = (105.5\%) \times \$160.00 = 1.055 \times \$160.00 = \$168.80 \; .$
Here, the final answer was rounded up to the next penny.

New car prices of a particular make went from $13,499 to $14,200 . What was the percent increase?
Solution: $\text{Percent Increase} = (14{,}200 - 13{,}499) \div 13{,}499 \times 100\% = 5.19\% \; .$

When a paint is cooled from 75°F to 10°F the viscosity (as measured with a viscosity cup) increases 89% . At 10°F the time for the paint to drain from the cup is 49 seconds. How long did it take the paint to drain at 75°F ?
Solution: An 89% increase in viscosity means that the low temperature viscosity is 100% + 89% = 189% of the high temperature viscosity.

$$\text{Drain Time at } 10°\text{F} = 189\% \times \text{Drain Time at } 75°\text{F} = 1.89 \times \square = 49\,\text{sec}$$

$$\Rightarrow \square = 49\,\text{sec} \div 1.89 = 25.9\,\text{sec} \; .$$

When a measurement is specified on a precision part, such as a diameter of 0.150 in plus or minus 0.001 in , the 0.001 in is called the tolerance. This means that any measured diameter between 0.149 in and 0.151 in is acceptable. Another way of specifying the tolerance is the percent tolerance defined by the following formula :

$$\text{Percent Tolerance} = \frac{\text{tolerance}}{\text{specified value}} \times 100\% \quad .$$

Again multiplying by 100% changes the fractional tolerance into a percent tolerance. The following examples illustrate percent tolerance.

A resistor is rated at $85\,\Omega \pm 2\,\Omega$. What is the percent tolerance?

Solution: $\text{Percent Tolerance} = 2\,\Omega \div 85\,\Omega \times 100\% = 2.35\%$.

An acceptable voltage reading is 15.0 V with a percent tolerance of 3% . Express this tolerance in volts.

Solution: $\text{Tolerance} = 3\% \times 15.0\text{ V} = 0.03 \times 15.0\text{ V} = 0.45\text{ V}$.

So the voltage is $15.0\text{ V} \pm 0.45\text{ V}$.

Percent error is similar to percent tolerance. When an actual measurement is made and compared to the "true" or "specified" value, the percent error is defined by the following formula :

$$\text{Percent Error} = \frac{\text{measured value} - \text{specified value}}{\text{specified value}} \times 100\% \quad .$$

Usually if the calculated percent error is negative the minus sign is ignored. This is because we are concerned with "how close" we are to the specified value and not whether we are above or below it. The following illustrates a percent error calculation.

A part is specified as having a diameter of 0.250 in . The manufactured part measures 0.254 in . What is the percent error?

Solution: $\text{Percent Error} = \left(0.254\text{ in} - 0.250\text{ in}\right) \div 0.250\text{ in} \times 100\% = 1.60\%$.

Some sales personnel earn part or all of their salary on a commission basis. This means that they are paid a percentage of their total sales. For example, consider the following situation.

A sales person earns $250 per week plus a 2.5% commission. If the person desires a week's gross pay of $900, how much merchandise must be sold?

Solution: The amount of commission = $900 – $250 = $650 .

$2.5\% \text{ of sales} = 0.025 \times \text{sales} = \$650 \Rightarrow \text{sales} = \$650 \div 0.025 = \$26{,}000$.

When getting a loan from a lender, we must pay back both the amount borrowed (the **principal**) plus **interest** for the temporary use of the lender's money. In a simple interest loan everything is paid in one lump sum. For example, if you borrow $1500 at an interest of 5.0%, then you must pay back 100% of the principal plus 5.0% of the principal = 105% of the principal or (1.05)($1500) = $1575 .

Most conventional loans are paid back in a sequence of monthly payments. In each payment a portion is used to pay the interest owed on the remaining debt and what is left over is used to reduce the debt, i.e., is paid against principal. A conventional loan is characterized by the following four parameters.

1. The initial principal symbolized by P.

2. The annual percentage rate (APR) of interest symbolized by R.

3. The number of years over which the loan is paid off (the period of amortization) symbolized by N .

4. The monthly payment symbolized by M.

To calculate M knowing P, R, and N use the formula: $$M = \frac{P \cdot R}{12 \cdot \left[1 - 1 \div (1 + R/12)^{12 \cdot N}\right]} .$$

To calculate N knowing P, R and M use the formula: $$N = \frac{\log\left[1 - \frac{R \cdot P}{12 \cdot M}\right]}{-12 \cdot \log\left[1 + \frac{R}{12}\right]} .$$

To calculate P knowing M, R, and N use the formula: $$P = \frac{12 \cdot M}{R} \cdot \left[1 - 1 \div (1 + R/12)^{12 \cdot N}\right] .$$

In the formula for N, log is the logarithm function, which is found in the second row, fifth column of the Casio *fx-300MS* and the first row, third column of the TI-30Xa. These formulas are rather complicated although they can be easily computed using a spreadsheet program such as Excel. With patience they can also be done on a scientific calculator. To illustrate these calculations consider the following three examples.

1. If you borrow $12,000 at an annual percentage rate of 2.3% to be paid off over 4 years, what is your monthly payment?

Solution: P = $12,000, R = 0.023 and N = 4, so we need to use the formula for M. Because the number of instructions that can be stored on the TI-30Xa is less than that of the Casio *fx-300MS*, the keystrokes for the TI-30Xa must be entered differently than on the Casio *fx-300MS*. On the Casio *fx-300MS* calculate M with the following keystrokes:

[( ] P [×] R [ )] [÷] [( ] 12 [×] [( ] 1 [−] 1 [÷] [( ] 1 [+] R [÷] 12 [ )] [^] [( ] N [×] 12 [ )] [ )] [ )] [=] . For the values given for P, R, and N this works out to be $261.92 , if rounded up to the next penny. Remember that calculators older than the Casio *fx-300MS* use the symbol x^y for raising a base to an exponent instead of ^ .

The corresponding keystrokes for the TI-30Xa are

[( ] 1 [−] 1 [÷] [( ] 1 [+] R [÷] 12 [ )] [y^x] [( ] N [×] 12 [ )] [ )] [1/*x*] [×] P [×] R [÷] 12 [=] . The [1/*x*] key is in the third row, second column.

2. If you borrow $18,000 at an annual percentage rate of 3.1% and make monthly payments of $324.24, how long does it take to pay off the loan?

Solution: P = $18,000, R = 0.031 and M = $324.24, so we need to use the formula for N.

On the Casio *fx-300MS* calculate N with the following keystrokes :

[log] [( ] 1 [−] R [×] P [÷] [( ] 12 [×] M [ )] [ )] [÷] [( ] [(−)] 12 [×] [log] [( ] 1 [+] R [÷] 12 [ )] [ )] [=] . For the values given for P, R, and M the computed value for N is 4.99949 or 5 years.

The corresponding keystrokes for the TI-30Xa are

[( ] 1 [−] R [×] P [÷] [( ] 12 [×] M [ )] [ )] [log] [÷] [( ] 12 [+⇄−] [×] [( ] 1 [+] R [÷] 12 [ )] [log] [ )] [=] .

3. If you can afford monthly payments of $350 over a three year period, what is the most money you can borrow at an annual percentage rate of 4.2%?

Solution: M = $350, R = 0.042 and N = 3, so we need to use the formula for P.

On the Casio *fx-300MS* calculate P with the following keystrokes :

[( ] 12 [×] M [÷] R [ )] [×] [( ] 1 [−] 1 [÷] [( ] 1 [+] R [÷] 12 [ )] [^] [( ] N [×] 12 [ )] [ )] [=] . Plugging in the values given for M, R, and N we calculate that P, if rounded up to the next penny, is $11,819.12 . On a practical level you can afford to borrow about $11,800 .

The corresponding keystrokes for the TI-30Xa are

[( ] 1 [−] 1 [÷] [( ] 1 [+] R [÷] 12 [ )] [y^x] [( ] N [×] 12 [ )] [ )] [×] 12 [×] M [÷] R [=] .

Exercises:

Write each of the following numbers as a percent :

0.37 1) ______________

0.012 2) ______________

2.59 3) ______________

$\frac{3}{8}$ 4) ______________

$1\frac{2}{3}$ 5) ______________

Write each percent as a decimal number :

22.5% 6) ______________

211% 7) ______________

Write each percent as a fraction reduced to lowest terms :

125% 8) ______________

$1\frac{5}{6}$ % 9) ______________

Solve for the following unknowns:

What is 16.5% of 128?

10) ______________

125% of $950 is what amount?

11) ______________

53 is what percent of 120?

12) ______________

12 is to 80 like 45 is to what number?

13) ______________

12 is to 60 like what amount is to 90?

14) ______________

$125 is 36% of what amount?

15) ______________

What percent of $\frac{3}{8}$ is $\frac{5}{16}$?

16) ______________

A license cost $120. If the cost increases 7.5%, what is the new cost of this license?

17) ______________

If sales tax is 5.5%, what would be the check out price of a band saw with a list price of $289? (Round to the nearest penny.)

18) ______________

From a 15.0 lb cylinder 1.6 lb of material is removed during machining. What percent of material was removed?

19) ______________

An electrical resistor is rated at 85 ohms plus or minus 5%. Express this tolerance in ohms .

20) ______________

A piston is to have a diameter of 0.787 in ± 0.003 in . What is the percent tolerance?

21) ______________

Specifications call for a pin to be 1.500 in long. If the finished pin measures 1.504 in., what is the percent error?

22) ______________

If an electric drill usually selling for $89.95 is on sale at a discount of 25%, what is the new list price? (Round to the nearest penny.)

23) ______________

A new car is advertised as selling for $14,220. This price reflects a 9% discount. What was the original (list) price? (Round to the nearest penny.)

24) ______________

An assembly line is shut down for inspection if the fraction of defective products exceeds 0.5% . If the normal day's production is 10,500 units, at most how many defective units can there be if the line is not to be shut down?

25) ______________

A salesperson is paid $380 per week plus a 2.2% commission. What is the person's sales total if the gross pay for a given week is $1,395? (Round to the nearest penny.)

26) ______________

What is the monthly payment required to pay off a $12,000 loan in two years at an annual percentage rate of 2.7% ? (Round to the nearest penny.)

27) ______________

What is the largest amount which can be borrowed over three years at 4.5% APR if the largest affordable monthly payment is $279? (Round to the nearest ten dollars.)

28) ______________

How long would it take to pay off $15,000 at 5.2% APR if the monthly payment is $450? (Round to the nearest tenth of a year.)

29) ______________

You can afford 15% of your monthly income of $2300 on car payments. If the quoted annual percentage rate of the loan is 2.5% over three years, what is the most you can borrow? (Round to the nearest ten dollars.)

30) ______________

Answers:

1. 37% ; 2. 1.2% ; 3. 259% ; 4. 37.5% ; 5. 166.67% ; 6. 0.225 ; 7. 2.11 ; 8. $1\frac{1}{4}$; 9. $\frac{11}{600}$

10. 21.12 ; 11. $1187.50 ; 12. 44.17% ; 13. 300 ; 14. 18 ; 15. $347.22 ; 16. 83.33%

17. $129 ; 18. $304.90 ; 19. 10.67% ; 20. $85\Omega \pm 4.25\Omega$; 21. $\pm 0.38\%$; 22. 0.27%

23. $67.46 ; 24. $15,626.37; 25. 52 ; 26. $46,136.36 ; 27. $514.18 ; 28. About $9380

29. 3.0 years ; 30. About $11,950

Measurement

All measurements consist of two parts, a number part and a unit. For example, if we measure a table with a tape measure and report its width as 37.5 in , 37.5 is the number part, in is the unit part. The connection between the number part and unit part is the operation of multiplication. So when we write 37.5 in , we really mean (37.5)(1 in) , i.e., 37.5 multiples of the base unit of one inch.

There are many different kinds of things we can measure, but all common physical measurements can be reduced to just a few kinds of things. The four basic quantities are often listed as follows :

1. Length measured in units such as feet, miles, meters, etc.

2. Time measured in units such as seconds, minutes, hours, etc.

3. Mass (or weight) measured in units such as grams, kilograms, pounds, etc.

4. Electric charge measured in coulombs or amp-seconds.

All other measurements can be expressed as a combination of these. It should really be noted that mass and weight are not identical. Weight is the gravitational force acting on a mass and varies with position in space. For example, the weight of an object on the surface of the earth is about six times the weight of that same object on the surface of the moon. However, the mass of the object is the same in both locations. Despite this difference, in these notes we will not distinguish between mass and weight.

It is not sensible to add or subtract measurements of different kinds of things. For example, 15 lb + 7 ft is a meaningless operation. This is just the old adage that it's impossible to add apples and oranges! To add or subtract measurements requires the same kind of quantities, as in 9 ft + 8 ft = 17 ft . Note, we just add the numbers and carry the factor of the unit. This is just the distributive property discussed in Unit 3. Suppose we have 8ft + 36 in . Here, the quantities to be added are both lengths so the operation makes sense, but we can't actually perform the addition until we get the units to agree as in 8 ft + 36 in = 8 ft + 3ft = 11 ft . Here we "converted" 36 in to 3 ft . The operation could also have been done by converting 8 ft to 72 in , and adding 72 in + 36 in = 108 in .

While adding or subtracting different kinds of measurements is impossible, multiplying or dividing measurements is always possible. For example,

$5\text{ lb}\times 4\text{ ft} = 20\text{ ft}\cdot\text{lb}$, where a $\text{ft}\cdot\text{lb}$ is $1\text{ ft}\times 1\text{ lb}$ is a "foot pound", a measure of either energy or torque.

As a second example,

$100\text{ miles} \div 4\text{ gallons} = 25\text{ mpg}$ (miles per gallon), which measures fuel economy .

Measurement conversion is a necessary skill since the same set of units is often not used throughout a calculation. The basis of measurement conversion is the unit fraction. Any quantity Q remains unchanged when multiplied by 1.

$$Q = Q \cdot 1$$

The catch is that 1 has infinitely many "aliases". For example,

$$1 = \frac{12\text{ in}}{1\text{ ft}} = \frac{1\text{ ft}}{12\text{ in}} = \frac{4\text{ qt}}{1\text{ gal}} = \frac{1\text{ gal}}{4\text{ qt}} = \frac{60\text{ sec}}{1\text{ min}} = \frac{1\text{ min}}{60\text{ sec}} \quad .$$

All of these represent unit fractions, since the numerator is the same amount as the denominator. The trick is to use the "proper" aliases to cancel the units you don't want and get the units you do want. For example, to convert 18 in to ft we could use the following procedure.

$$18\text{ in} = \frac{18\text{ i}\not{n}}{1} \times \frac{1\text{ ft}}{12\text{ i}\not{n}} = \frac{18}{12}\text{ ft} = \frac{3}{2}\text{ ft} = 1.5\text{ ft} \quad .$$

Here the in unit was cancelled by appearing in both numerator and denominator.

Consider rounding 0.434 in to the nearest 64'th of an inch. The trick is to use the unit fraction

$$\frac{64}{64}\text{ as follows}: 0.434\text{ in} \times \frac{64}{64} = 0.434 \times 64 \times \frac{1\text{ in}}{64} = 27.776 \times \frac{1\text{ in}}{64} \approx \frac{28}{64}\text{ in} = \frac{7}{16}\text{ in} \quad .$$

So 0.434 in is seven sixteenth's of an inch to the nearest 64'th of an inch.

More complicated conversions can involve more than one unit fraction. The speed 100 feet per second can be converted to miles per hour correct to 1 decimal place by the following :

$$100\frac{\text{ft}}{\text{s}} = 100\frac{\text{ft}}{\text{s}} \times \frac{60\text{ s}}{1\text{ min}} \times \frac{60\text{ min}}{1\text{ hr}} \times \frac{1\text{ mile}}{5280\text{ ft}}$$

$$100\frac{\text{ft}}{\text{s}} = 100\frac{\not{\text{ft}}}{\not{\text{s}}} \times \frac{60\not{\text{s}}}{1\text{ m}\not{\text{i}}\text{n}} \times \frac{60\text{ m}\not{\text{i}}\text{n}}{1\text{ hr}} \times \frac{1\text{ mile}}{5280\not{\text{ft}}} = \frac{100 \times 60 \times 60}{5280}\frac{\text{mile}}{\text{hr}} = 68.2\text{ mph} \quad .$$

As an aide in setting up conversion calculations a set of equivalent measurements is presented on the following page.

This list also includes the formulas for converting temperature from Fahrenheit ($^\circ$F) to Celsius ($^\circ$C) . For example, to find the Fahrenheit equivalent of 40°C , we calculate as follows :

$$\text{Temp }^\circ\text{F} = 40 \times \frac{9^\circ\text{F}}{5} + 32^\circ\text{F} = 104^\circ\text{F} \;,$$

while the Celsius equivalent of minus 10°F is computed using the formula

$$\text{Temp }^\circ\text{C} = (-10 - 32) \times \frac{5^\circ\text{C}}{9} = -23.3^\circ\text{C} \;.$$

Conversion Relations for English and Metric Units

Linear Measure :

1 ft = 12 in
1 yd = 3 ft
1 mile = 5280 ft
1 rod = 16.5 ft
1 furlong = 220 yd
1 in = 2.54 cm
1 ft = 0.3048 m
1 yd = 0.9144 m
1 mile = 1.609344 km

Area Measure :

1 acre = 160 sq rods
1 sq mile = 640 acres
1 cm^2 = 0.15500031 in^2
1 m^2 = 1.195990046 yd^2
1 km^2 = 0.3861021585 sq mile

Volume Measure :

16 oz = 1 pt
2 pt = 1 qt
4 qt = 1 gal
1 gal = 0.13368056 ft^3
1 gal = 231 in^3
1 gal = 3.78541178 L
1 ft^3 = 7.48051948 gal
1 ft^3 = 28.31684659 L
1 L = 0.26417205 gal
1 L = 1.056688209 qt
1 L = 61.02374409 in^3
1 L = 0.001 m^3
1mL = 1 cm^3

Weight Measure :

16 oz = 1 lb
1 oz = 28.348 g
1 ton = 2000 lb
1 lb = 453.568 g
1 kg = 2.20474 lb

Temperature Conversions :

$$\text{Temp}^\circ\text{F} = \text{Temp}^\circ\text{C} \times \frac{9}{5} + 32^\circ\text{F}$$

$$\text{Temp}^\circ\text{C} = \left(\text{Temp}^\circ\text{F} - 32\right) \times \frac{5}{9}$$

Time Measure :

1 min = 60 s
1 hr = 60 min
1 hr = 3600 s
1 day = 24 hours

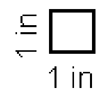
There is a definite relationship between length, area and volume measurements. Area is the amount of “two-dimensional space” inside of a planar figure. For example, a 2 ft by 6 ft rectangle has an area, A = (2 ft)(6 ft) = 12 ft^2. Here, the unit ft^2 is one square foot (sq ft) which literally means a one foot by one foot square. When we say that the area of the rectangle is 12 ft^2, we mean that we could fit exactly 12 one foot by one foot squares inside this rectangle.

Care must be taken when converting units of area. Suppose we want to calculate how many square inches are in an area of 1.6 ft^2. We need the unit fraction between square inches and square feet.

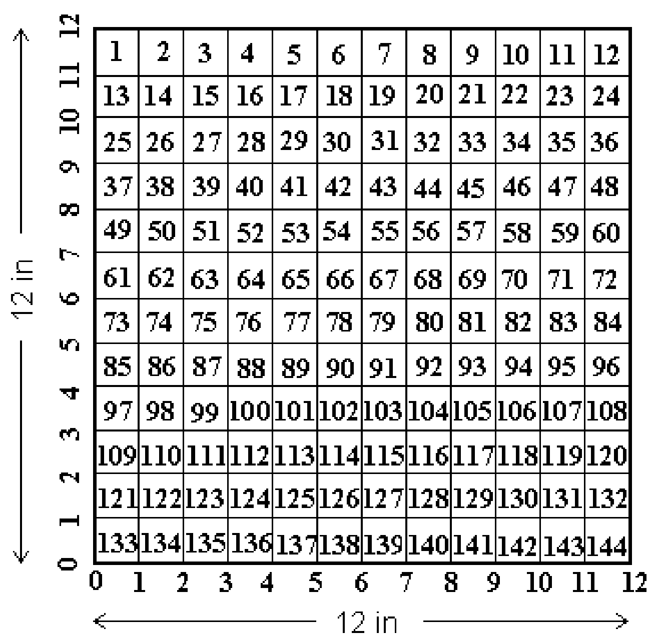
$$1\,\text{ft}^2 = (12\,\text{in})^2 = 12\,\text{in} \times 12\,\text{in} = 144\,\text{in}^2.$$

Note : when we evaluate $(12\ \text{in})^2$ we square **both** the 12 **and** the in . This is illustrated below.

1 square inch =



1 square ft=144 square in =

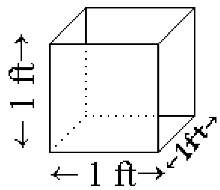


To perform the conversion,

$$1.6\,\text{ft}^2 = 1.6\,\cancel{\text{ft}}^2 \times \frac{144\,\text{in}^2}{1\,\cancel{\text{ft}}^2} = 1.6 \times 144\,\text{in}^2 = 230.4\,\text{in}^2.$$

Volume is the amount of “three-dimensional space” inside of a solid. For example, a 2 ft by 3 ft by 2 ft box has a volume, V = (2 ft)(3 ft)(2 ft) = 12 ft^3**.** Here the unit ft^3 is one cubic foot (cu ft), which literally means a one foot by one foot by one foot cube as shown below.

$$1ft^3 = 1ft \times 1ft \times 1ft =$$



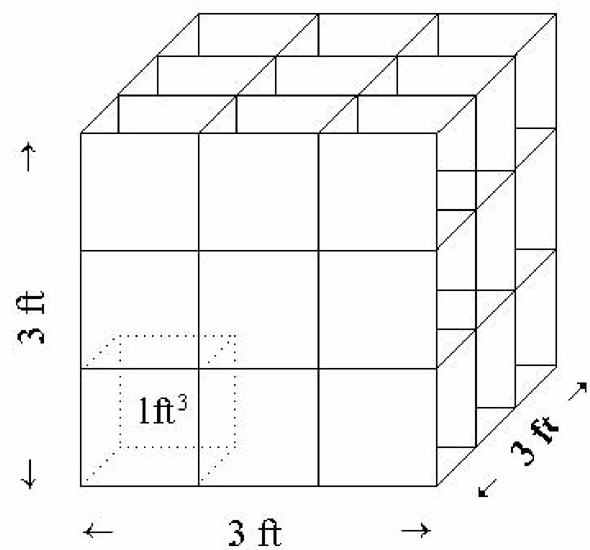
When we say that the volume of the box is 12 ft^3, we mean that we could fit exactly 12 one foot by one foot by one foot cubes inside this box. Like area conversions, volume conversions require careful setup. Suppose we wish to convert 12 ft^3 to cubic yards.

$$1\,\text{yard} = 3\,\text{ft}$$

$$1\,\text{cu yd} = 1\,\text{yd}^3 = (3\text{ft})^3 = 3\text{ft} \times 3\text{ft} \times 3\text{ft} = 27\,\text{ft}^3.$$

Note : when we evaluate $(3\text{ ft})^3$ we cube **both** the 3 **and** the ft . This is illustrated below.

$$1yd^3 = 3ft \times 3ft \times 3ft = 27ft^3 =$$



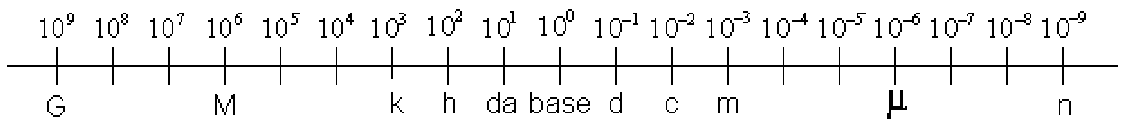
To perform the conversion,

$$12\,\text{ft}^3 = 12\,\text{ft}^3 \times \frac{1\,\text{yd}^3}{27\,\text{ft}^3} = \frac{12}{27}\,\text{yd}^3 = 0.44\text{yd}^3.$$

One of the consequences of the French Revolution of 1789 was the development of the metric system of measurement. This system was designed to replace the earlier French system, which like its English counterpart had its origins in medieval society and royal institutions. Three features make the metric system very attractive. First, it is built on powers of 10, just like our decimal number system. Every unit is a multiple of 10 of some other unit. Thus, “strange” English multipliers like 3, 12, and 16 are banished! Second, a deliberate effort was made to coordinate different measures. For example, the fundamental unit of volume, the liter symbolized by L, is simply related to the fundamental unit of length, the meter symbolized by m, through the equation $1\ m^3 = 1000\ L$. Contrast this with the English system where $1\ gal = 231\ in^3 = 0.134\ ft^3$. The third advantage of the metric system is that it is “universal”. It can be used with **any** kind of measurement in the same way.

It is interesting to note, that the metric system was so well accepted and in place that when electrical measurements began some 150 years ago only metric units were developed and have survived. The customary electric units we are all know, the volt (V), amp (A), and ohm (Ω) are all metric.

The metric system uses a two-part representation of all measurements. The first character or prefix indicates the power of 10 used, while the remainder of the measurement is the base unit. This is illustrated below.



$10^{-9} = 0.000000001$
$10^{-8} = 0.00000001$
$10^{-7} = 0.0000001$
$10^{-6} = 0.000001$
$10^{-5} = 0.00001$
$10^{-4} = 0.0001$
$10^{-3} = 0.001$
$10^{-2} = 0.01$
$10^{-1} = 0.1$
$10^{0} = 1$
$10^{1} = 10$
$10^{2} = 100$
$10^{3} = 1000$
$10^{4} = 10,000$
$10^{5} = 100,000$
$10^{6} = 1,000,000$
$10^{7} = 10,000,000$
$10^{8} = 100,000,000$
$10^{9} = 1,000,000,000$

Prefix Names

G=giga
M=mega
k =kilo
h =hecto
da =deca
d=deci
c=centi
m=milli
µ=micro
n=nano

Important Metric Base Units

length - meter abbreviated m
time - second abbreviated s
mass - gram abbreviated g
volume - liter abbreviated L
(Note: the metric system has been set up so 1 mL= 1 cm^3)
power - watt - abbreviated w
energy - joule - j
force - newton -N
frequency - hertz abbreviated Hz
electric current - ampere abbreviated A
electric potential - volt abbreviated V
electric resistance - ohm abbreviated Ω
electric capacitance - farad abbreviated F
electric inductance - henry abbreviated H

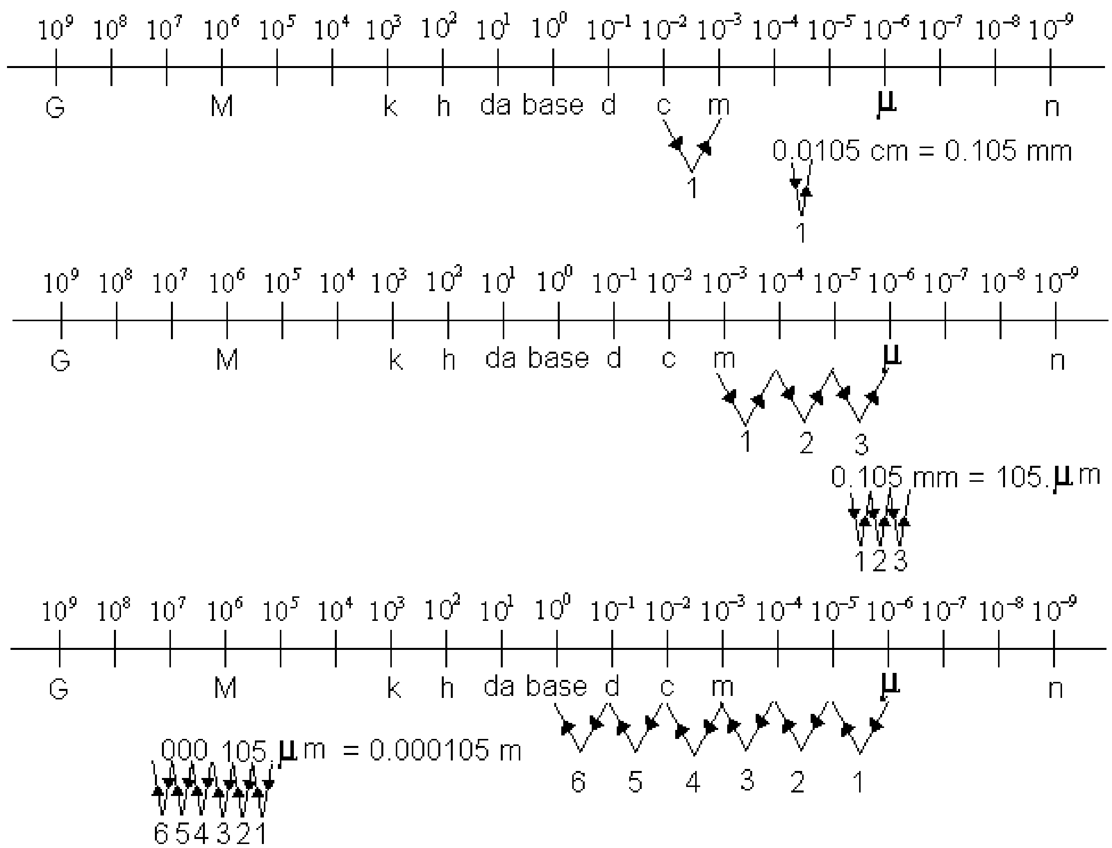
Conversions within the metric system are particularly easy. The steps are as follows :

1. Lay out a chart as shown below.

2. Locate the starting unit position and the final unit position on this chart and note the direction from the starting unit to the final unit.

3. Count the number of positions on the chart from the starting unit space to the final unit space.

4. Move the decimal point of the number part of the measurement the same number of decimal places as the count in Step 3 and in the same direction as noted in Step 2.



As before area and volume conversions within the metric system require careful setup. For example, suppose we want to convert 0.042 m^2 to square cm. The calculation can be setup as follows :

$$0.042\,\text{m}^2 = 0.042\,\text{m}^2 \times \left(\frac{100\,\text{cm}}{1\,\text{m}}\right)^2 = 0.042\,\cancel{\text{m}}^2 \times \frac{10000\,\text{cm}^2}{1\,\cancel{\text{m}}^2} = 0.042 \times 10000\,\text{cm}^2 = 420\,\text{cm}^2.$$

As second example, to convert 187 mm^3 to mL, we proceed as shown below :

$$187\,\text{mm}^3 = 187\,\text{mm}^3 \times \left(\frac{1\,\text{cm}}{10\,\text{mm}}\right)^3 \times \frac{1\,\text{mL}}{1\,\text{cm}^3} = 187\,\cancel{\text{mm}}^3 \times \frac{1\,\cancel{\text{cm}}^3}{1000\,\cancel{\text{mm}}^3} \times \frac{1\,\text{mL}}{1\,\cancel{\text{cm}}^3}$$

$$= \frac{187}{1000}\,\text{mL} = 0.187\,\text{mL}.$$

Conversions between metric and English units require conversion factors. For example, to convert 1.80 gallons per minute to m^3 per hour, we can use the following procedure :

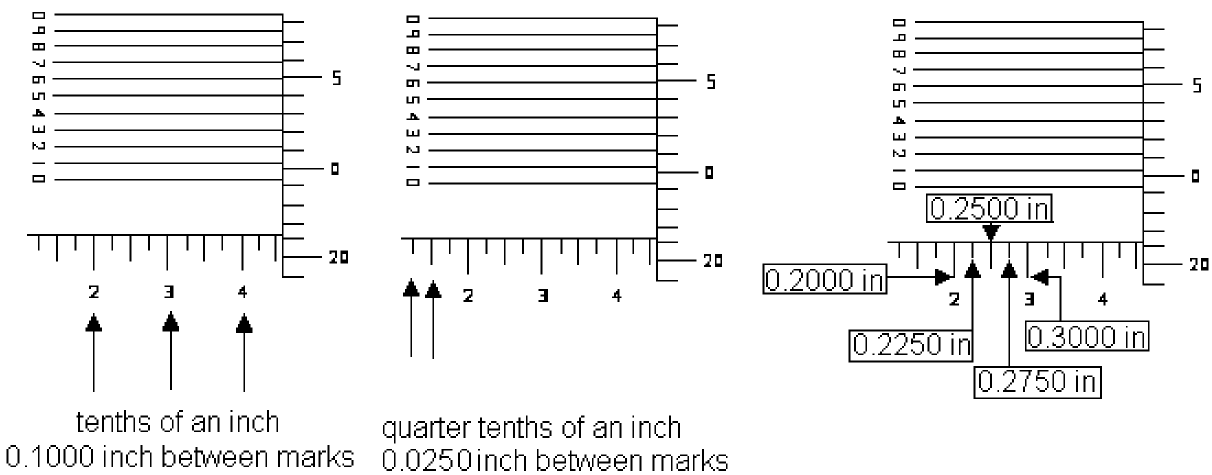
$$1.80\,\frac{\text{gal}}{\text{min}} = 1.80\,\frac{\cancel{\text{gal}}}{\cancel{\text{min}}} \times \frac{60\,\cancel{\text{min}}}{1\,\text{hr}} \times \frac{3.7854\,\cancel{\text{L}}}{1\,\cancel{\text{gal}}} \times \frac{0.001\,\text{m}^3}{1\,\cancel{\text{L}}} = 1.80 \times 60 \times 3.7854 \times 0.001\,\frac{\text{m}^3}{\text{hr}}$$

$$= 0.409\,\frac{\text{m}^3}{\text{hr}}.$$

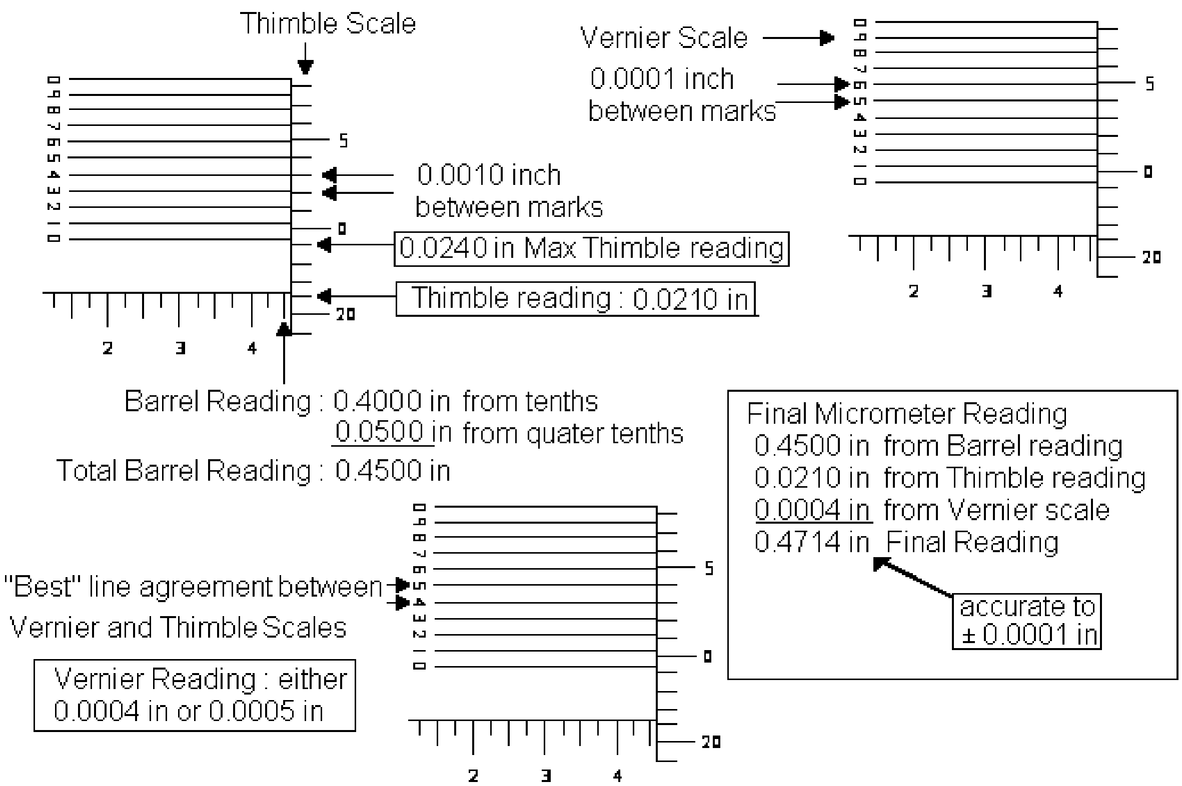
Length measurements accurate to the nearest 0.05 in can be made with a English decimal rule. English Vernier calipers can give measurements accurate to the nearest 0.01 in . To achieve higher accuracy an instrument called a micrometer (pronounced my-crom-a-ter) is used. To avoid confusion with metric terminology a measurement of 1 μm is often called a micron instead of a micrometer (pronounced my-crow-meter). Common English micrometers can be used to measure lengths to the nearest 0.001 in . English Vernier micrometers are ten times more accurate since they can measure to the nearest 0.0001 in .

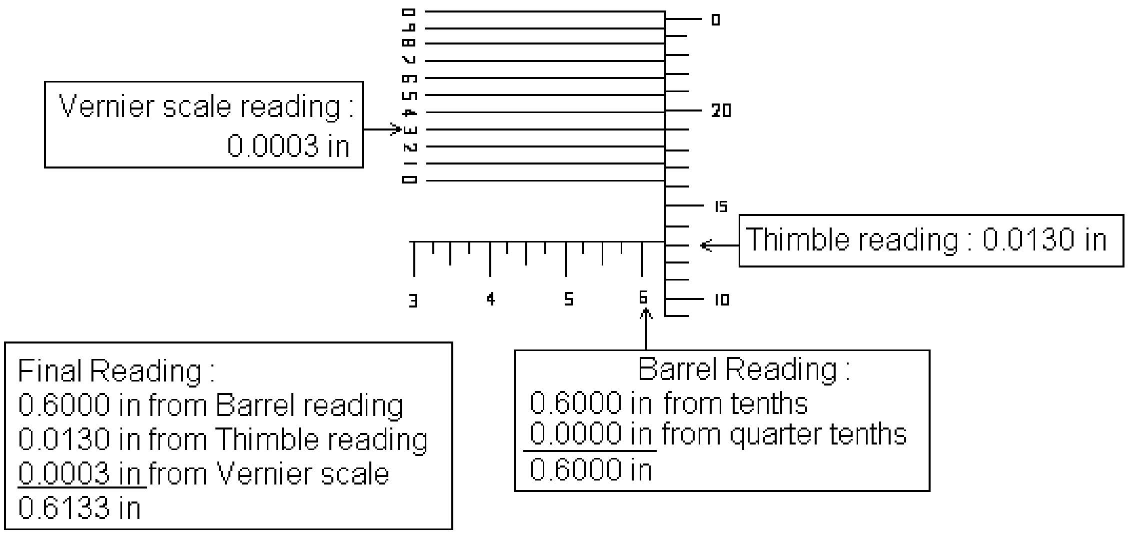
By placing an object between the “jaws” of the micrometer (called the anvil and the spindle) and turning the thimble until the anvil and spindle just make contact, we obtain the length of the object by combining the barrel, thimble, and Vernier scale readings. This is illustrated on the following pages.

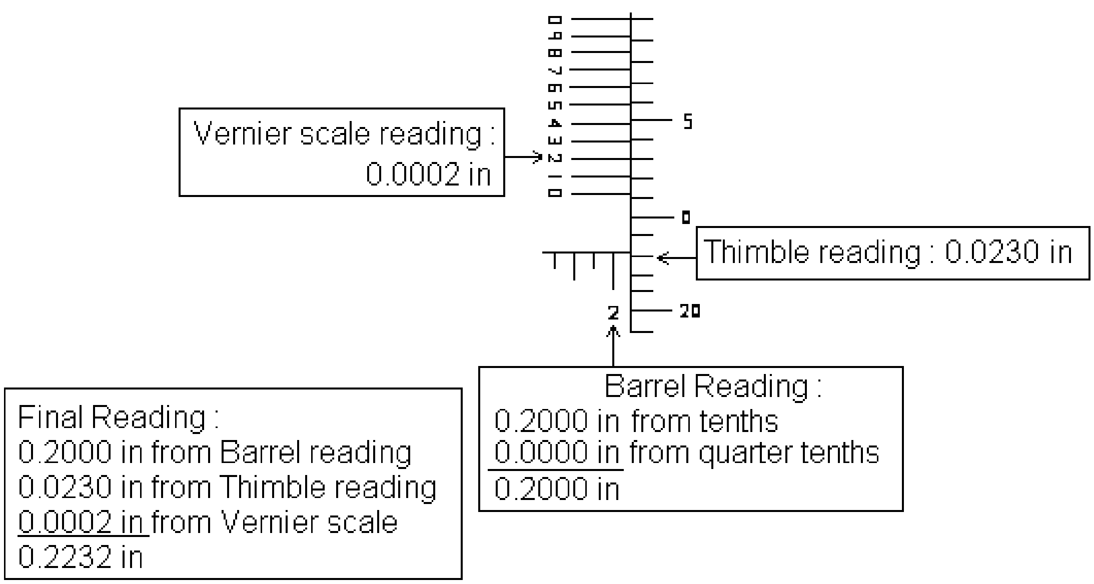
Reading an English Vernier Micrometer: Barrel Scales



Reading an English Vernier Micrometer: Thimble and Vernier Scales



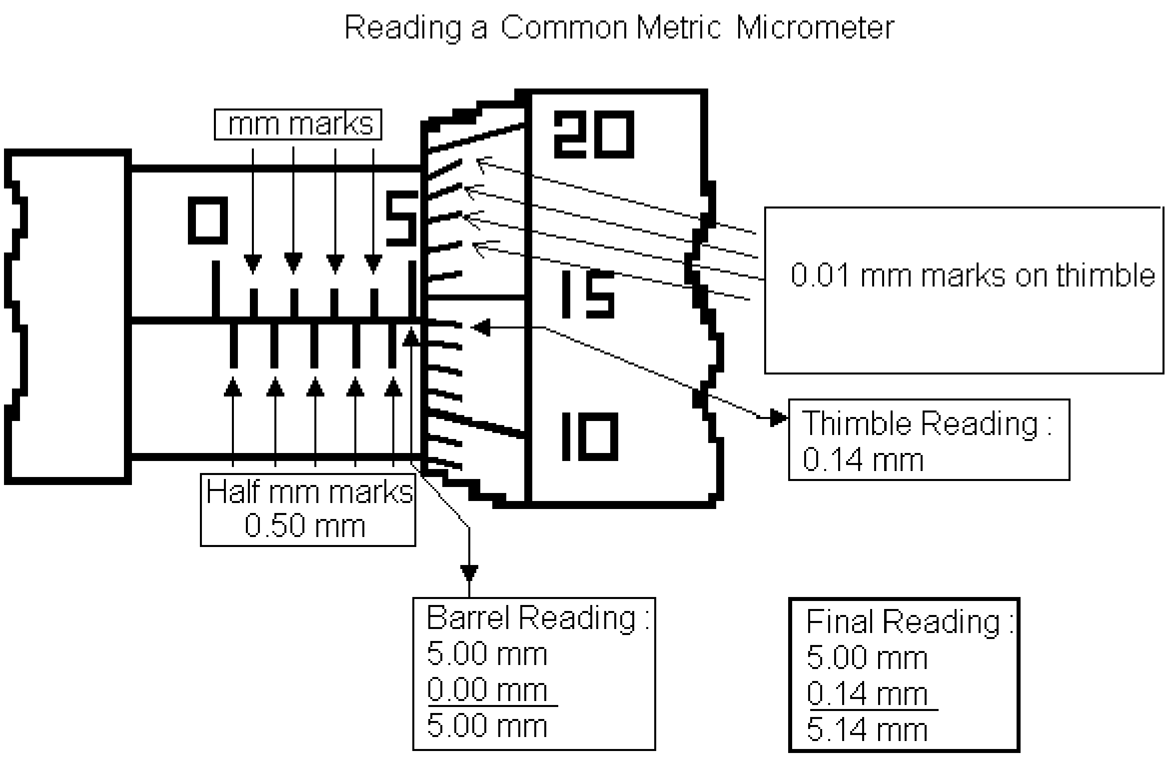

0 9 8 7 6 5 4 3 2 1 0
0
20
15
10
3 4 5 6
Vernier scale reading : 0.0003 in
Thimble reading : 0.0130 in
Barrel Reading :
0.6000 in from tenths
0.0000 in from quarter tenths
0.6000 in
Final Reading :
0.6000 in from Barrel reading
0.0130 in from Thimble reading
0.0003 in from Vernier scale
0.6133 in

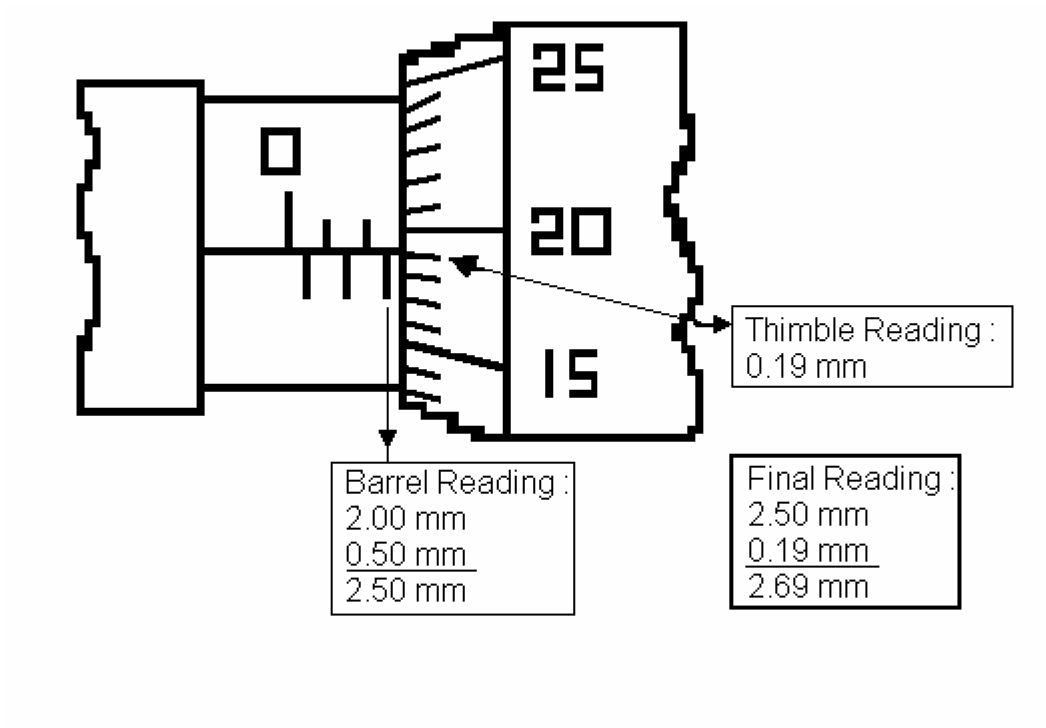

0 9 8 7 6 5 4 3 2 1 0
5
0
20
2
Vernier scale reading : 0.0002 in
Thimble reading : 0.0230 in
Barrel Reading :
0.2000 in from tenths
0.0000 in from quarter tenths
0.2000 in
Final Reading :
0.2000 in from Barrel reading
0.0230 in from Thimble reading
0.0002 in from Vernier scale
0.2232 in

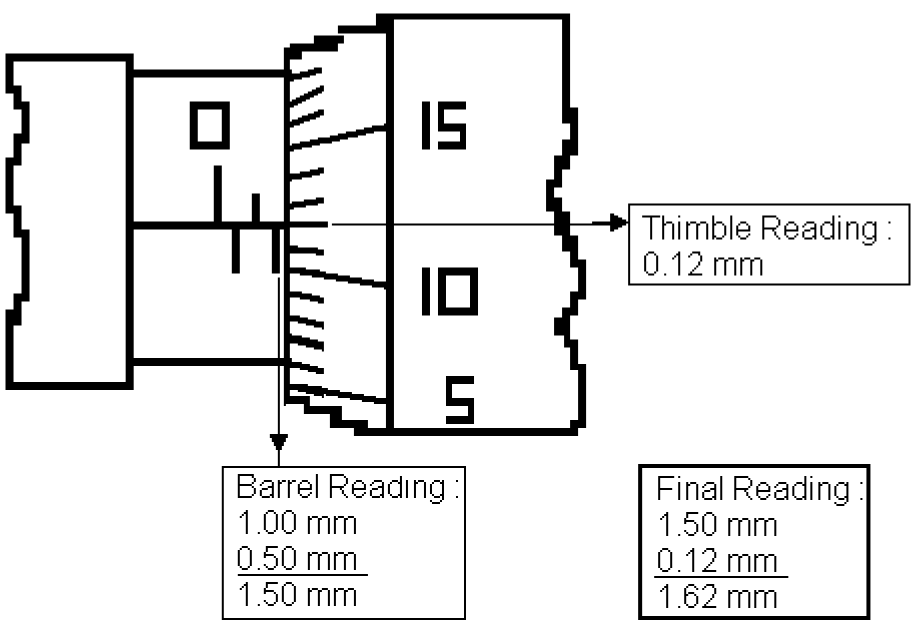
A common metric micrometer is accurate to the nearest

$$0.01\,\text{mm} = 0.01\,\cancel{\text{mm}}. \times \frac{1\,\text{in}}{2.54\,\cancel{\text{cm}}} \times \frac{1\,\cancel{\text{cm}}}{10\,\cancel{\text{mm}}} = 0.00039\,\text{in}\ .$$

So it is more accurate than a common English micrometer, but not as accurate as an English Vernier micrometer. The reading of a common metric micrometer is accomplished by combining the barrel and thimble readings. This is illustrated in the following figures.




25
0
20
15
Thimble Reading :
0.19 mm
Barrel Reading :
2.00 mm
0.50 mm
2.50 mm
Final Reading :
2.50 mm
0.19 mm
2.69 mm


15
0
10
5
Thimble Reading :
0.12 mm
Barrel Reading :
1.00 mm
0.50 mm
1.50 mm
Final Reading :
1.50 mm
0.12 mm
1.62 mm

Exercises:

Perform the following calculations with measurement numbers. Round to one decimal place.

12 ft 3 in − 8 ft 8 in

1) ______________

$4.2\,\text{cm} \times 3.5\,\text{cm}^2$

2) ______________

$530\,\text{miles} \div 35\,\text{mpg}$ (mpg = miles per gallon)

3) ______________

Convert the following measurements as indicated. Write answers in the blank space provided. Round to one decimal place.

4. $185\,^\circ$F = ______________ $^\circ$C

5. 17.6 liters = ______________ gal

6. 132 kg = ______________ lb

7. 127 ft^3 = ______________ yd^3

8. 129 mA = ______________ A

9. 0.589 m = ______________ mm

10. 58 cm = ______________ m

11. 127µV = ______________ V

12. 98 km/hour = ______________ mph

13. 10.0 $^\circ$C = ______________ $^\circ$F

14. 2.56 sq ft = ______________ cm^2

15. 2.49 gal = ______________ mL

16. $179cm^3$ = ______________ in^3

Round to the nearest 32'nd of an inch: 0.165 in =

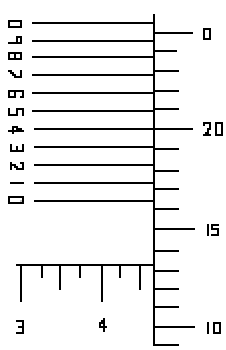
17) ______________

Round to the nearest 64'th of an inch: 0.645 in =

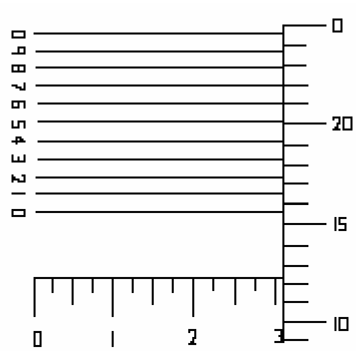
18) ______________

Read the following measuring devices.

English Vernier micrometer

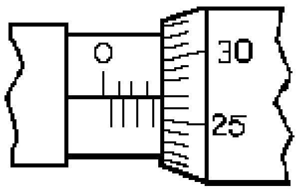


19) ______________

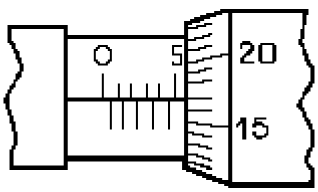


20) ______________

Common metric micrometer



21) ______________



22) ______________

What size bolt, to the nearest 64'th of an inch, will fit a hole 12 mm in diameter?

23) ______________

A box has dimensions of 5 ft 6 in $\times$ 3 ft 9 in $\times$ 3 ft 4 in . How many cubic meters is this?

24) ______________

A car has a gas tank with a capacity of 50 L. If the car gets 33.5 miles per gallon, how many km can the car travel on a full tank?

25) ______________

Answers :

1. 43 in = 3 ft 7 in ; 2. 14.7 cm^3 ; 3. 15.1 gal ; 4. 85°C ; 5. 4.65 gal ; 6. 291 lb ; 7. 4.70 yd^3

8. 0.129 A ; 9. 589 mm ; 10. 0.58 m ; 11. 0.000127 V ; 12. 61 mph ; 13. 50°F

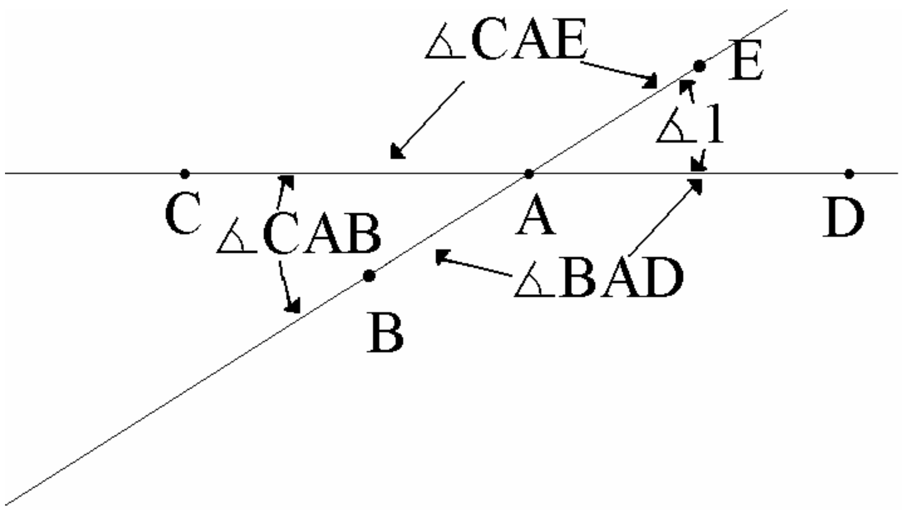
14. 2378 cm^2 ; 15. 9426 mL ; 16. 10.9 in^3 ; 17. $\frac{5}{32}$ in ; 18. $\frac{41}{64}$ in ; 19. 0.4634 in

20. 0.3126 in ; 21. 3.77 mm ; 22. 5.17 mm ; 23. $\frac{15}{32}$ in ; 24. 1.95 m^3 ; 25. 712 km

Applied Geometry

The word geometry in Greek means "earth measurement" and had its origins in problems associated with surveying. In these notes we are considering plane or flat two-dimensional geometry. (Solid geometry studies objects in three dimensions.) The fundamental notions used to describe planar objects are points and lines. Two different points define one and only one line that passes through both of them and the shortest distance between the two points is along this line. We shall use capital letters such as **A** to label points and a pair of letters such as **AB** to label the segment of the line between **A** and **B** . Line segments are measured in units of length such as feet or meters .

Two lines in a plane either intersect at a single point or are parallel. Where the lines meet four "openings" or angles are formed as shown below. The symbol ∡ is used for the word angle.



In order to label angles several conventions are used. As shown in the above picture, the point where the lines intersect **A** is called the **vertex** of the four angles. The separate angles are then named by indicating a point on each of the two lines which act as sides of the angle. Thus ∡**BAD** is the angle with vertex at **A** and with sides that include segments **AB** and **AD** . This same angle could also be labeled as ∡**DAB** . A less precise but more convenient notation is to label angles by a single letter or number, provided that the symbol is placed appropriately in the diagram. For example, in the above diagram ∡**1=**∡**EAD** . The most commonly used measure for angles is the degree system developed by the Babylonians over 3000 years ago. They decided for reasons of easy divisibility by whole numbers to divide a circle into 360 equal units of angle called a degree and signified by the symbol °. For precise surveying or problems in astronomy a smaller unit is required. This lead to the minute which is 1/60'th of a degree and the second which is 1/60'th of a minute. The symbol for minute is ′ , while the symbol for second is ″.

Thus, we have $1' = \frac{1}{60}^\circ$ and $1'' = \frac{1}{60}' = \frac{1}{60} \times \frac{1}{60}^\circ = \frac{1}{3600}^\circ$. An angle to the nearest second could be specified as $\measuredangle 4 = 37°29'54''$, this notation is called DMS for Degree Minute Second. To avoid fractions this same angle could be given in DD (Decimal Degrees) as

$\angle 4 = (37 + \frac{29}{60} + \frac{54}{3600})° = 37.4983333...°$. We could also go from DD to DMS as follows :

$$\angle 4 = 37.4983333...° = 37° + \frac{60'}{1°} \times .498333...° = 37°29.9' = 37°29' + .9' \times \frac{60''}{1'} = 37°29'54''$$

These conversions have been built into many scientific calculators. For example, to enter $37°29'54''$ in DMS notation on the Casio *fx-300MS* , keystroke

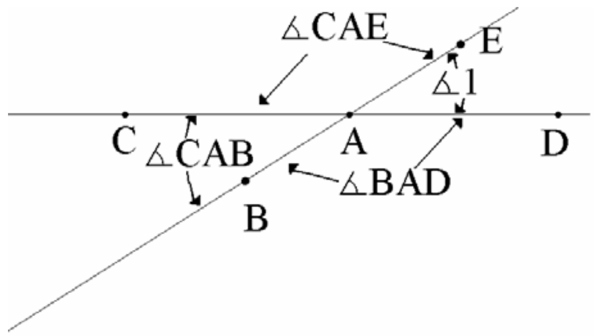
37 [° ′ ″] 29 [° ′ ″] 54 [° ′ ″] , where [° ′ ″] is the key in the third row second column. The display will then show the DD result of 37.49833333°. To convert back to DMS notation keystroke [SHIFT | ° ′ ″] and the calculator will display 37°29°54 which is the Casio's way of indicating $37°29'54''$. To enter $37°29'54''$ using the TI-30Xa, keystroke

37.29.54 [2nd | +] , where the second decimal point (which does **not** show in the display) separates the seconds from the minutes

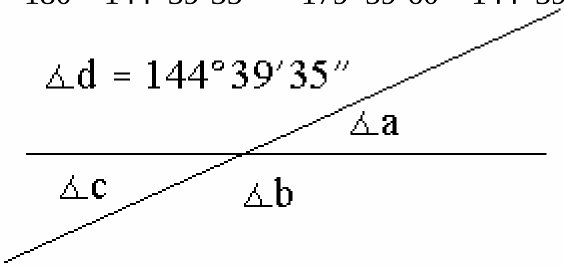
and [2nd | +] is the [DMS to DD] function. To convert from DD to DMS use

[2nd | =]. These conversions can be imbedded into calculations involving angular measure.

Since there are 360° in a circle, the angular measure of a straight line is 180°, while in a square or right angle there are 90°. Two angles whose sum is 180° make a straight line when combined and are called **supplementary**. Consider again the intersection of two lines.
$\measuredangle$**EAD** + $\measuredangle$**CAE**=$\measuredangle$**EAD** + $\measuredangle$**BAD** = 180°. Since $\measuredangle$**EAD** is on both sides of this equation, we conclude that $\measuredangle$**CAE** = $\measuredangle$**BAD** and by a similar argument that $\measuredangle$**CAB**=$\measuredangle$**EAD**. These equal angles formed by two intersecting lines are called **vertex angles**.



As an application of this result consider the following diagram. Once the measure of △**d** is known, the remaining three can be determined. Since they are vertex angles,
△**b** = △**d** = 144°39′35″. Since △**a** is the supplement to △**d,**
△**a** = 180°–144°39′35″ = 179°59′60″–144°39′35″= 35°20′25″ = △**c** . This same calculation



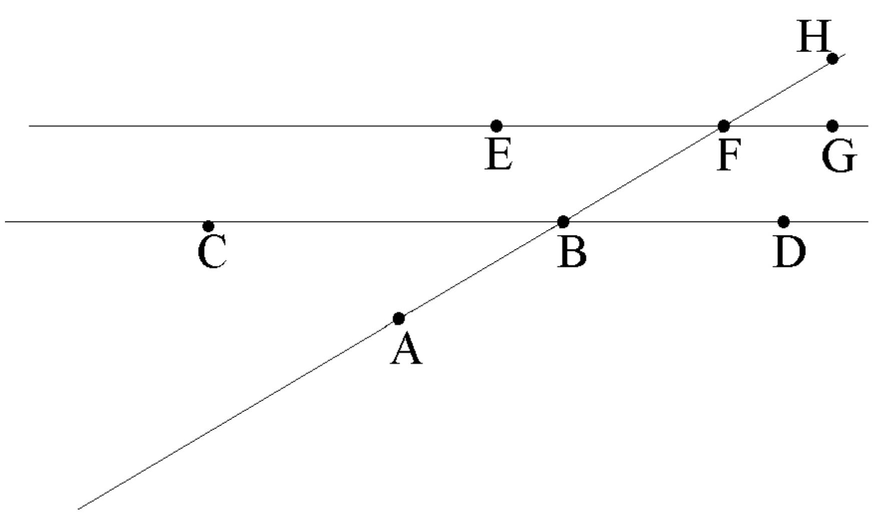
is done on the Casio *fx-300MS* using the following keystrokes:

180 [−] 144 [° ′ ″] 39 [° ′ ″] 35 [° ′ ″] [=] [SHIFT] [° ′ ″]

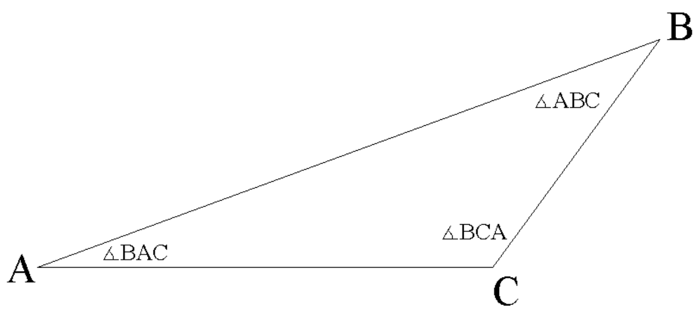
and on the TI-30Xa using

180 [−] 144.39.35 [2nd] [+] [=] [2nd] [=] .

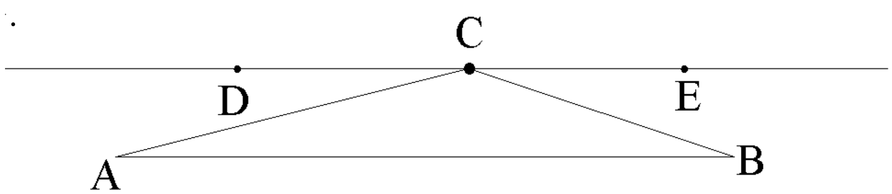
Consider a pair of parallel lines crossed by a third line (called the **transversal**) as shown below. If we imagine that point **B** is superimposed over point **F** by moving segment **CD** onto the line through **EG** , the **corresponding** angles △**EFB** and △**CBA** are equal. Similarly, △**EFH=**△**CBF** , △**ABD=**△**BFG** , and △**FBD=**△**HFG** . From the equality of vertex angles, the **alternating interior** (alternate sides of the transversal, inside the two parallel lines) are equal, i.e., △**EFB=**△**FBD** and △**CBF=**△**BFG** . Similarly, the **alternating exterior** (alternate sides of the transversal, outside the two parallel lines) are equal, i.e., △**EFH=**△**ABD** and △**CBA=**△**HFG** .

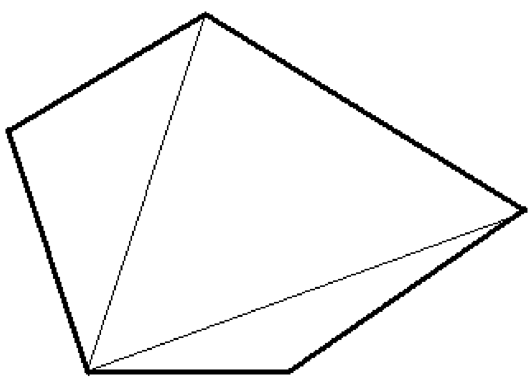


Polygons are closed figures in the plane whose sides are line segments. The simplest polygon is the three-sided **triangle.** The points at the corners **A** , **B** , and **C** are the **vertices** of the triangle, and the angles △**BAC** , △**BCA** , and △**ABC** are called the interior angles of the triangle. The triangle is often then labeled as triangle **ABC** .



Consider the triangle **ACB** shown below. Through the vertex **C** we construct a line parallel to the base segment **AB.** Since they are alternating interior angles, △**DCA** = △**CAB** and △**ECB** = △**CBA** . However, △**DCA** + △**ACB** + △**ECB** = 180°, so we have △**CAB** + △**ACB** + △**CBA** = 180° or in words, the interior angles of a triangle always sum to 180° .

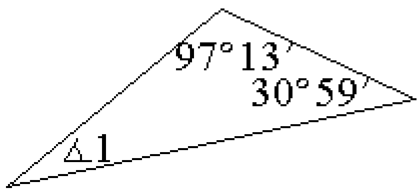




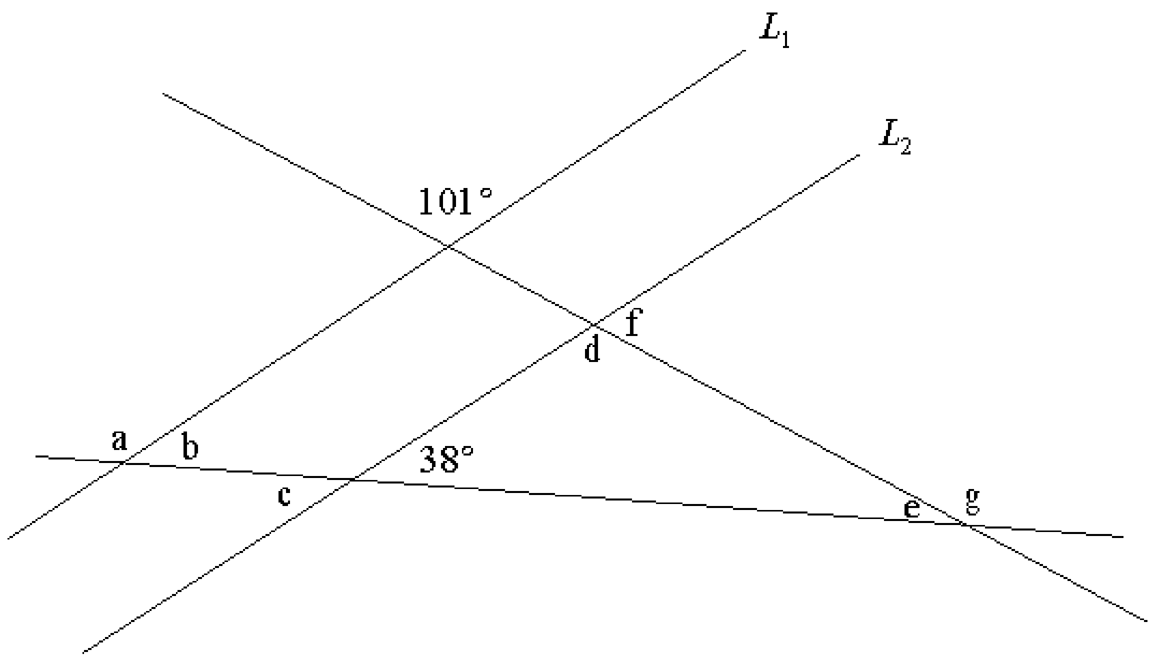
Consider the five-sided pentagon on the right. By drawing three triangles from one of the vertices, we see that the sum of the internal angles of the pentagon is the sum of all the internal angles in the three triangles or 3(180°) = 540°. A similar argument for an n-sided polygon shows that the sum of the internal angles is (n–2)(180°).

As an application of these principals, consider the following problems. In the triangle below the missing angle $\triangle\mathbf{1}$ is calculated as follows :

$$\triangle\mathbf{1} = 180° - 97°13' - 30°59' = 51°48'$$



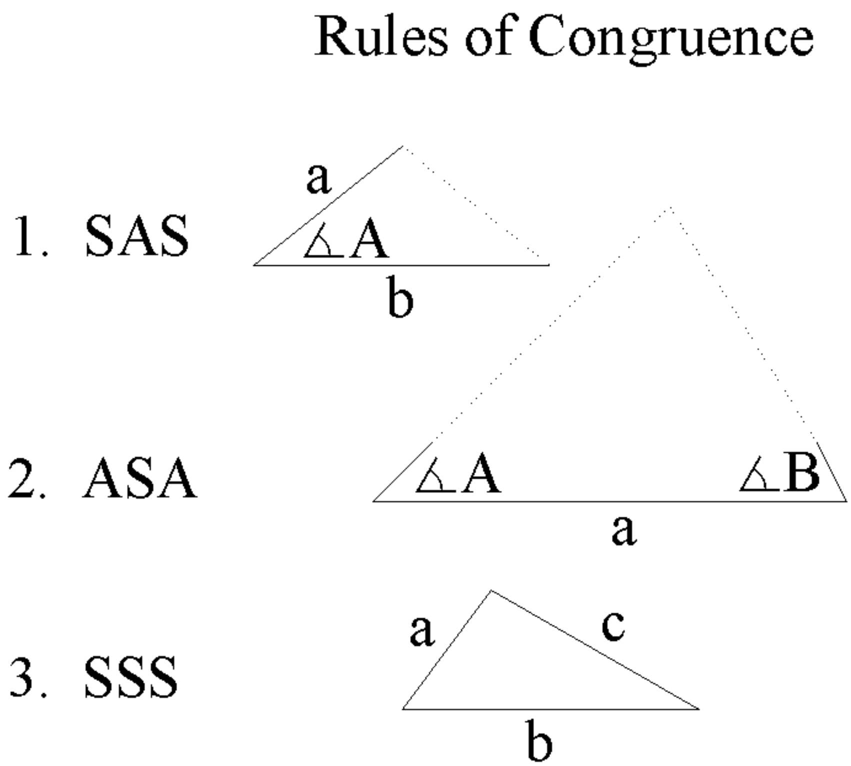
If L_1 and L_2 are parallel, then $\triangle\mathbf{d}$ =101°, $\triangle\mathbf{f}$ =180°– $\triangle\mathbf{d}$ = 79°, $\triangle\mathbf{b}$ = $\triangle\mathbf{c}$ = 38°, $\triangle\mathbf{a}$ =180°– $\triangle\mathbf{b}$ = 142°, $\triangle\mathbf{e}$ =180°– $\triangle\mathbf{d}$ –38°= 41°, and $\triangle\mathbf{g}$ =180°– $\triangle\mathbf{e}$ = 139°.



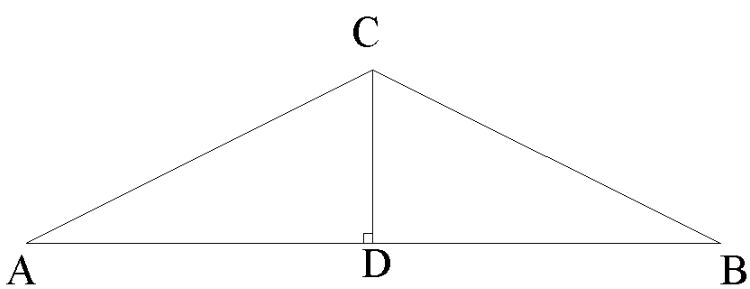
We now turn our attention to triangles. The first question we need to consider is what determines a triangle. Every triangle has three sides and three angles, so six numbers (three angle measures and three side lengths) are associated with every triangle. Two triangles are called **congruent** if one can be superimposed exactly on the other. Essentially, triangles that are congruent are the "same". There are three rules for congruency : **SAS**, **ASA** and **SSS** .

The notation **SAS** means two sides and the angle between these two sides have been specified; the remaining side and two angles are determined. The notation **ASA** (or equivalently **AAS**, since if any two angles in a triangle are known, the third can be determined from180°minus the sum of the other two) means two angles and the side between them have been specified. Of course, the sum of the two specified angles must be less than 180°. Finally, **SSS** means all three sides have been specified. Since the shortest distance between any two vertices is the side joining

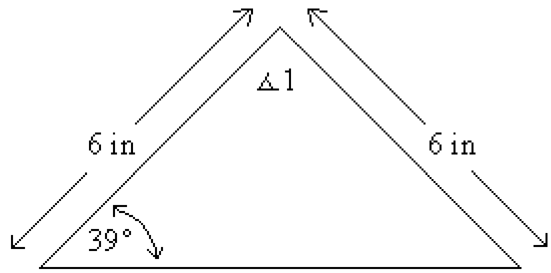
these points, the longest of the three sides of a triangle must be shorter than the sum of the other two sides for the triangle to exist.
Note : AAA , while it determines a triangle's shape, does not determine its size and is therefore **not** a rule of congruence.



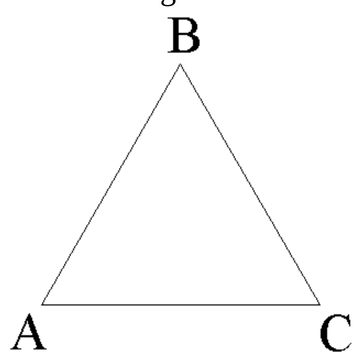
A triangle is called isosceles if two sides are equal . Consider the isosceles triangle **ACB** with **AC** = **CB** , from **C** construct the segment **CD** to **D** the mid point (i.e., **AD** = **DB**) of **AB** . Now by SSS triangle **ADC** is congruent to triangle **BDC** . Thus, ∡**ADC** = ∡**BDC**, and since these two angles sum to$180°$, **CD** is perpendicular (makes a right angle) to **AB** . This is indicated in the diagram by the "little" box at **D** . Also ∡**CAB** = ∡**CBA** ; in words , the angles opposite to the equal sides are also equal.



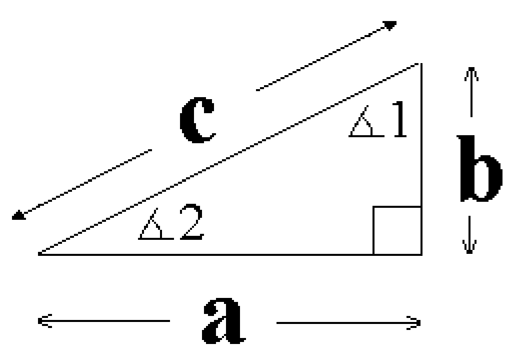
Consider the following isosceles triangle with $\measuredangle$**1** unspecified. Since the angles opposite the equal sides must be equal, $\measuredangle$**1** =180°– 2(39°) = 102°.



A triangle is called **equilateral** if all three sides are equal. Of course, equilateral triangles are also isosceles, so **AB = BC** and the opposite angles $\measuredangle$**BAC** and $\measuredangle$**BCA** are equal. But **AC** = **BC** , so $\measuredangle$**ABC** and $\measuredangle$**BAC** are equal . Thus all three angles are equal and since they sum to 180°, we have that for an equilateral triangle all of the internal angles equal 60°**.**



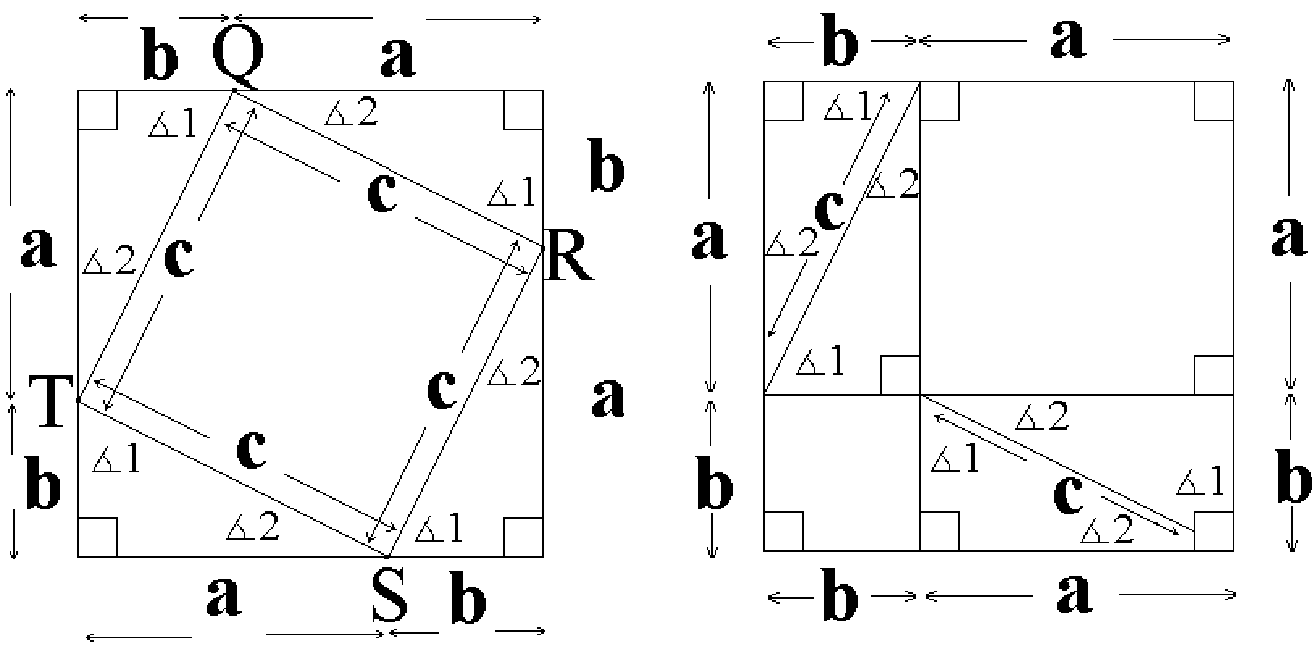
A **right triangle** is a triangle with a 90°interior angle. The large side opposite the 90°angle is called the **hypotenuse** while the remaining two sides are called **legs**. In the diagram shown, the hypotenuse is **c** and the legs are **a** and **b** . The two **acute** (less than 90°) angles are $\measuredangle$**1** and $\measuredangle$**2** with $\measuredangle$**1** opposite to the side of length **a** and $\measuredangle$**2** opposite to the side of length **b** . The sum of the three interior angles is $\measuredangle$**1** +$\measuredangle$**2** + 90°= 180° , so we have the result that the two acute angles in a right triangle are **complementary**, i.e., $\measuredangle$**1** +$\measuredangle$**2** = 90° .



Construct a right triangle with legs **a** and **b** and hypotenuse **c** and $\angle$**1** opposite to the side of length **a** and $\angle$**2** opposite to the side of length **b** . Consider the two ways of constructing a square of side **a + b** shown below.

In the figure on the left we have $\angle$**1** + $\angle$**TQR + $\angle$2 =** 180°, but since $\angle$**1** +$\angle$**2** = 90° , we know that $\angle$**TQR** = 90°. A similar argument shows that $\angle$**QTS =** $\angle$**QRS =** $\angle$**RST =** $\angle$**TQR=** 90°. So the figure in the center on the left is the square of side **c** , i.e., the square of the hypotenuse. Thus, the area of the square of side **a + b** is the square of **c** plus the area of the four right triangles.

From the figure on the right we see that the area of the square of side **a + b** is the square of **a** plus the square of **b** plus the area of the same four right triangles. Since the area of the two figures must be the same we conclude that the square of the hypotenuse is the sum of the squares of the legs, or in symbols $c^2 = a^2 + b^2$.



This result is called the **Pythagorean Theorem** and is probably the most famous and useful result in geometry! The result can be stated a number of different ways. To calculate the hypotenuse knowing the lengths of the legs, we take the square root and get

$$c = \sqrt{a^2 + b^2} \quad .$$

To calculate a leg, say **a**, knowing the hypotenuse and the other leg, **b**, the formula is rearranged to

$$a = \sqrt{c^2 - b^2} \qquad \text{or} \qquad \text{leg} = \sqrt{\text{hypotenuse}^2 - \text{other leg}^2} \quad .$$

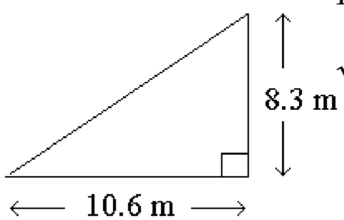
Remember the square root symbol is also a grouping symbol. Parantheses need to be used!

To find the missing hypotenuse of the following right triangle, we compute as follows :

$$c = \sqrt{\left((8.3\text{m})^2 + (10.6\text{m})^2\right)}$$

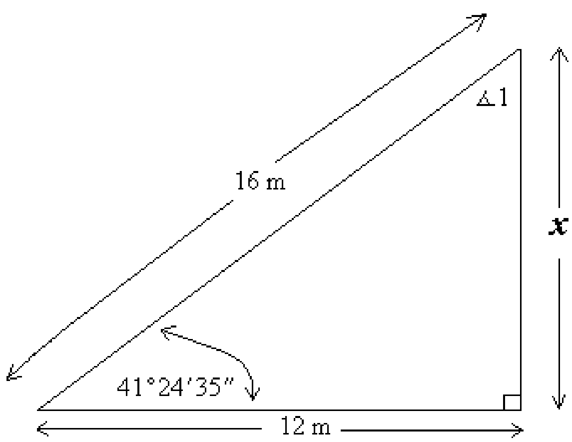
Note: The implied parenthesis inside the square root has been made explicit as required to get the correct answer on the calculator.

$$c = \sqrt{68.89\text{m}^2 + 112.36\text{m}^2} = \sqrt{181.25\text{m}^2} = 13.5\,\text{m}$$



The units work out to be linear units as required for a length since $\sqrt{\text{m}^2} = \text{m}$.

As a second example consider finding the unknown quantities in the following right triangle. The missing dimension *x* is a leg and can be calculated from

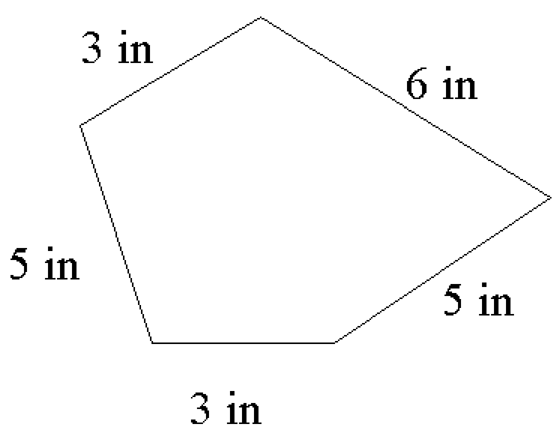


$$x = \sqrt{\left((16\text{m})^2 - (12\text{m})^2\right)} = \sqrt{112\text{m}^2} = 10.6\text{m}$$

The missing angle is complementary to $41°24'35''$, so

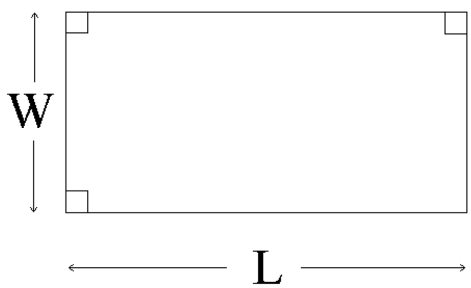
$$\angle 1 = 90° - 41°24'35'' = 48°35'25''$$

The last topic we consider is the **perimeters** and areas of figures in a plane. The perimeter is the total linear distance around the boundary of a polygon. For example, the perimeter of the pentagon shown below is computed as P = 3 in + 6 in + 5 in + 3 in + 5 in = 22 in .



Consider a rectangle of width W and length L. The perimeter is computed as

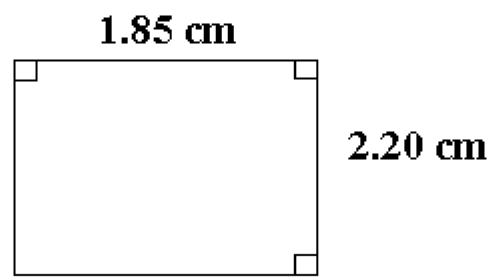
$$P = W + W + L + L = 2W + 2L$$



The area of a rectangle, which measures the amount of "two dimensional space" inside the rectangle is given by $A = W \cdot L$. This in fact is the **definition** of area. The area of all figures is based on this formula. The trick is to be clever enough to rearange the figure into a rectangle with the same area.

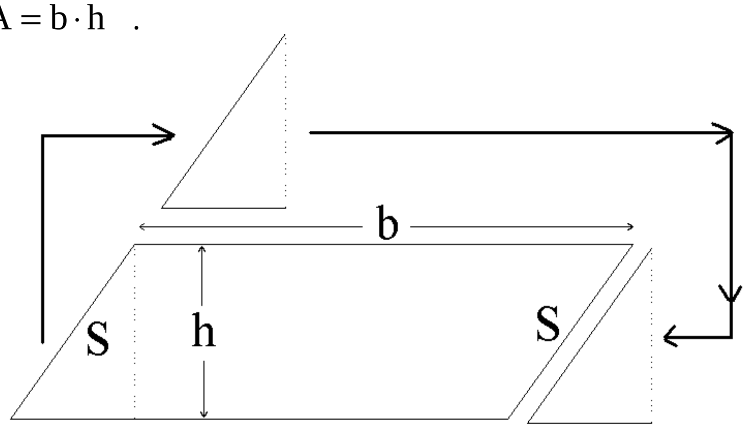
Note : perimeter always has units of length, while area always has units of length^2.
For the following rectangle, we compute P as P = 2(1.85 cm) + 2(2.20 cm) = 8.10 cm .

The area is given by $A = 1.85\,\text{cm} \times 2.20\,\text{cm} = 4.07\,\text{cm}^2$.

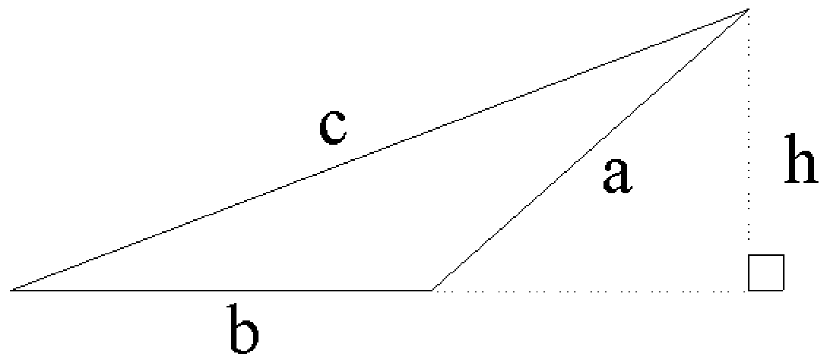


A **parallelogram** is a four sided polygon (or quadralateral) with opposite sides parallel. All rectangles are parallelograms, but in a generic parallelogram the angle between adjacent sides is not necessarily $90°$. Consider the parallelogram shown below with base **b** and perpendicular distance **h** between the top and bottom sides. Imagine cutting off a right triangle from the left end and moving it to the right end. Since the left and right sides are parallel, this right triangle fits perfectly to make a rectangle of dimensions **b** and **h** . So we have for a parallelogram that

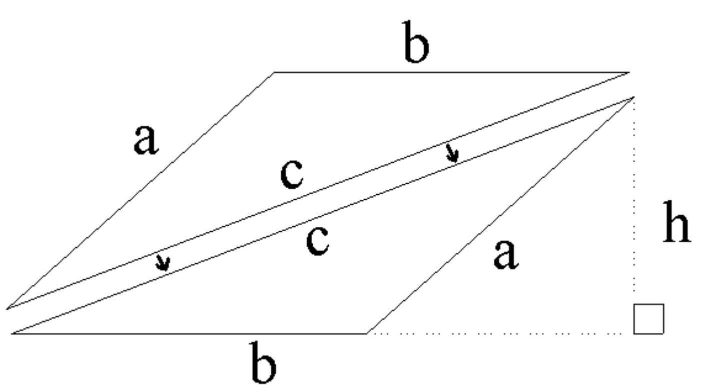
$$A = b \cdot h \quad .$$



Next consider a generic triangle triangle with bottom side (base) of length **b** and perpendicular distance (height) **h** from the base to the top vertex.



Imagine making an exact copy of this triangle and joining it to the original triangle as shown. The result is a parallelogram of base **b** and height **h** . Since the area of the original triangle is half of the area of this parallelogram, we arrive at the result that the area of a triangle is given by $A = \frac{1}{2} b \cdot h$.

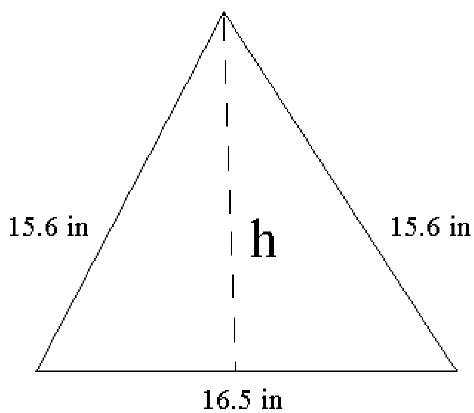


A more detailed argument shows that for a triangle with sides **a** , **b** , and **c** the area can be calculated from Heron's formula:

$$A = \sqrt{S(S-a)(S-b)(S-c)}$$.

with the semi-perimeter S given by the formula: $S = \frac{a+b+c}{2}$.

As an application consider calculating the perimeter and area of the following triangle. The perimeter is just the sum of the lenghts of the three sides.



$$P = 15.6\,\text{in} + 15.6\,\text{in} + 16.5\,\text{in} = 47.7\,\text{in}$$

To calculate the area, we could calculate the height by dropping a perpendicular from the top vertex. Since the triangle is isosceles, this bisects the 16.5 in base. Using the Pythagorean Theorem we compute h as follows :

$$h = \sqrt{(15.6\,\text{in})^2 - (16.5\,\text{in} \div 2)^2} = \sqrt{175.30\,\text{in}^2} = 13.2\,\text{in}$$

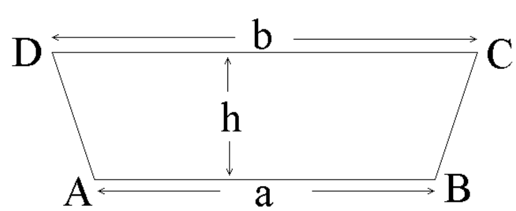
Then the area is calculated as half the base times the height.

$$A = 0.5 \times 16.5\,\text{in} \times 13.2\,\text{in} = 109\,\text{in}^2$$

Another way to calculate the area is to use Heron's formula. $S = \dfrac{47.7\,\text{in}}{2} = 23.85\,\text{in}$

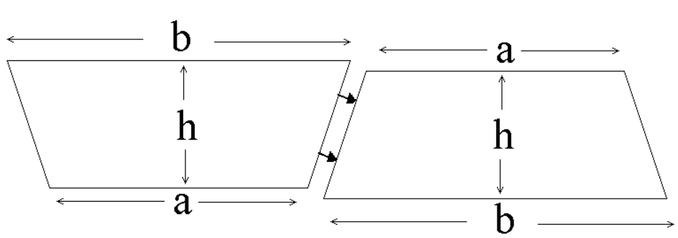
$$A = \sqrt{23.85\,\text{in}(23.85\,\text{in} - 15.6\,\text{in})(23.85\,\text{in} - 15.6\,\text{in})(23.85\,\text{in} - 16.5\,\text{in})} = \sqrt{11931\,\text{in}^4} = 109\,\text{in}^2$$

A quadrilateral with two opposite sides parallel is called a trapezoid. Suppose that the two parallel faces have lengths **a** and **b** and are separated by a perpendicular distance **h** .

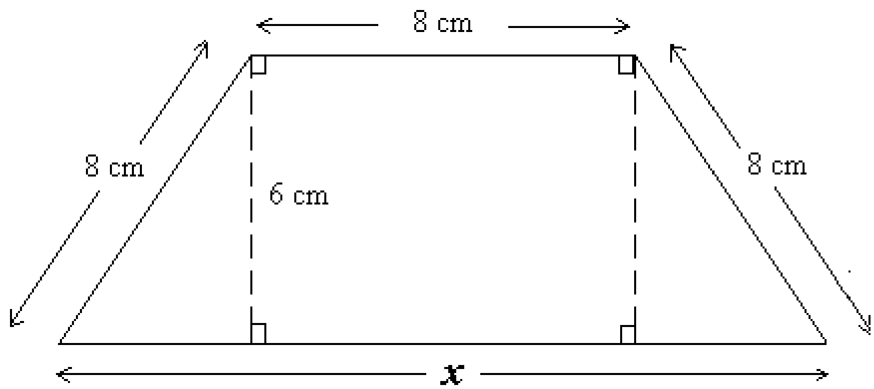


Imagine making an exact copy of this trapezoid and joining it to the original trapezoid as shown. The result is a parallelogram of base **a + b** and height **h** . Since the area of the original trapezoid is half of the area of this parallelogram, we arrive at the result that the area of a trapezoid is given by the folowing formula :

$$A = \frac{1}{2}(a + b) \cdot h \quad .$$



As an application consider calculating the perimeter and area of the following trapezoid.



The first step is to calculate the length x . Using the Pythagorean Theorem the length of the base of the right triangles that form the sides of the trapezoid is computed as follows :

$$\sqrt{(8\,\text{cm})^2 - (6\,\text{cm})^2} = \sqrt{28\,\text{cm}^2} = 5.29\,\text{cm} \ .$$

Then $x = 8\,\text{cm} + 2 \times 5.29\,\text{cm} = 18.58\,\text{cm}$. The perimeter is just the sum of the lenghts of the four sides. $\text{P} = 18.6\,\text{cm} + 8\,\text{cm} + 8\,\text{cm} + 8\,\text{cm} = 42.6\,\text{cm}$.

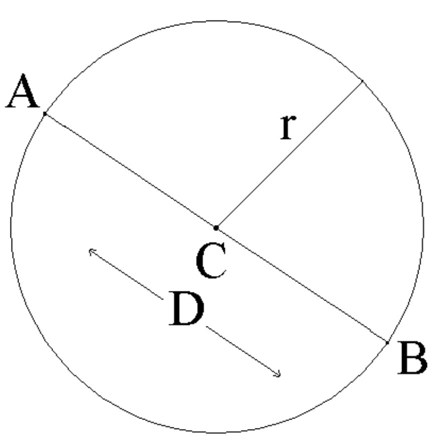
To calculate the area, we could add the area of the two right triangles that form the sides of the trapezoid to the area of the 8 cm by 6 cm central rectangle.

$$\text{A} = 2 \times \frac{1}{2} \times 5.29\,\text{cm} \times 6\text{cm} + 8\,\text{cm} \times 6\,\text{cm} = 79.7\,\text{cm}^2$$

We get the same result by using the formula for the area of a trapezoid .

$$\text{A} = \frac{1}{2} \times (18.58\,\text{cm} + 8\,\text{cm}) \times 6\text{cm} = 79.7\,\text{cm}^2$$

A circle is formed by generating all points in a plane which are a fixed distance called the **radius** from a center point here labeled as **C**. A line segment with end points on the circle that passes through the center is called a **diameter**. **D** and **r** symbolize the lengths of the diameter and radius , respectively.

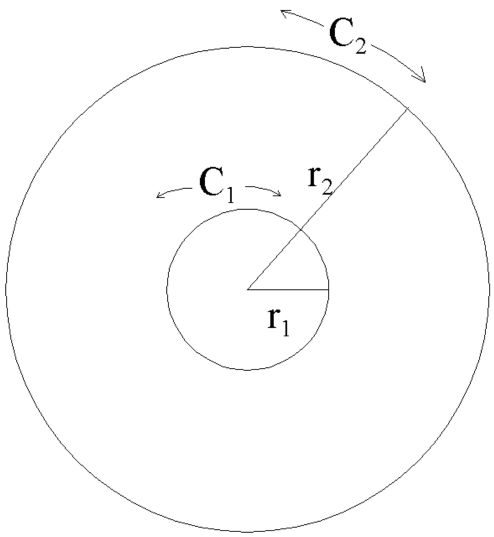


Since **AC = CB = r** and **D = AB = AC + CB =** 2**r** , We have the following formulas:

$$\text{D} = 2 \cdot \text{r}$$

$$\text{r} = \frac{\text{D}}{2}$$

All circles are **similar**. By this we mean that all circles have the same shape. If you've seen one circle, you've seen them all! Looked at another way, all circles are "scale models" of each other, and like in any scale model, the lengths of corresponding parts are in the same ratio . The distance around the boundary of a circle is called its **circumference** which is like the perimeter of a polygon. Imagine that we have two circles. The first labeled as **1** has radius $\mathbf{r_1}$, diameter $\mathbf{D_1}$ and circumference $\mathbf{C_1}$. The second labeled as **2** has radius $\mathbf{r_2}$, diameter $\mathbf{D_2}$ and circumference $\mathbf{C_2}$. Since all circles are scale models of each, the ratio of circumference to diameter is the same for both circles. The value of this ratio is symbolized by the Greek letter lower case pi.



$$\frac{C_2}{D_2} = \frac{C_1}{D_1} = \pi$$

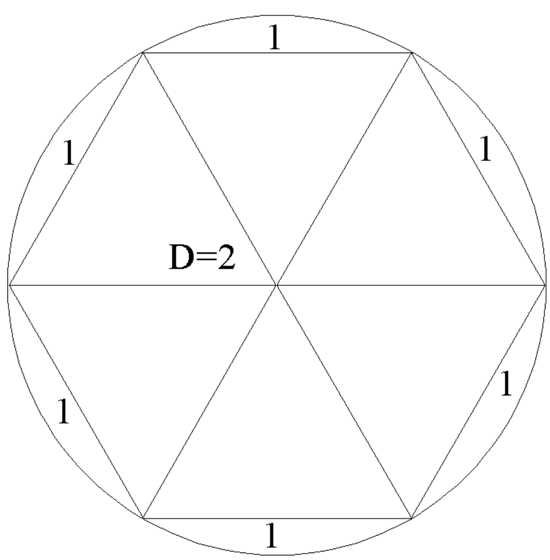
Rearranging this formula gives the results

$$C_2 = \pi \cdot D_2$$
$$C_1 = \pi \cdot D_1 ,$$

or more simply the relationship that circumference is pi times the diameter or twice pi times the radius.

$$C = \pi \cdot D = 2\pi \cdot r$$

The existence and value of pi strike many people as mysterious. However, a simple argument illustrated below shows that pi is slightly larger than 3. In fact the decimal approximation to pi is known to over a billion digits! There is a pi button on your calculator (eighth row, third column on the Casio *fx-300SA* ; third row, first column on the TI-30Xa). Pressing it gives $\pi \approx 3.141592654$.

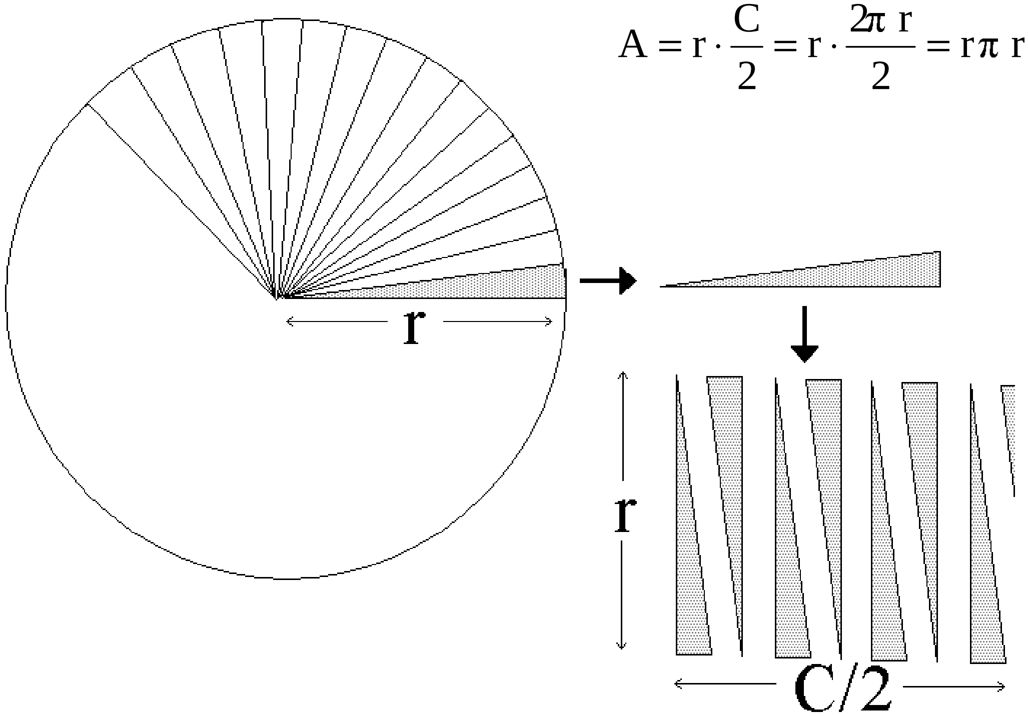


Form six equlateral triangles each of side 1 (the units of length don't matter). Since all of the internal angles equal to 60°, the six equilateral triangles can be joined next to each other with a common vertex as shown. The union of the six equilateral triangles is a regular **hexagon** with a perimeter of 6. Making the common vertex the center of a circle of radius 1, we see that the circumference of this circle is slightly bigger than the perimeter of the hexagon.

$$C > 6$$
$$\pi = C \div D > 3$$
$$C \approx 6$$
$$\pi \approx 6 \div D \approx 3$$

The determination of the formula for the area of a circle was done by Archimedes over 2200 years ago. To get some understanding of it, observe the following diagram. A circle is sliced up into an even number of thin slices. The slices "almost" look like triangles. The slices are then rearranged, half pointing up, the rest pointing down. Then they are pushed against each other until they just touch. The resulting figure resembles a rectangle, except the top and bottom have little bumps, but by making the number of slices bigger and bigger (this technique is called the method of "exhaustion"), the bumps get smaller and smaller until we do "get" a rectangle.
The height of the rectangle is **r** and the base is half of the circumference (since half of the slices faced up and the rest down). Thus we get the following "famous" formula for the area of a circle.

$$A = r \cdot \frac{C}{2} = r \cdot \frac{2\pi\, r}{2} = r\pi\, r = \pi\, r^2$$



As an application the circumference and area of a circle of diameter 2.500 in are computed by the following calculations :

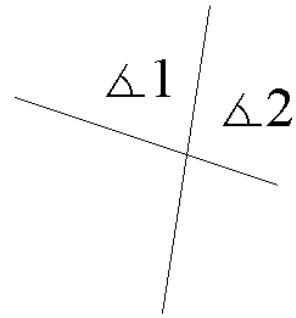
$$C = \pi \times 2.500\,\text{in} = 7.854\,\text{in}$$

$$r = \frac{D}{2} = \frac{2.500\,\text{in}}{2} = 1.250\,\text{in}$$

$$A = \pi \times (1.250\,\text{in})^2 = 4.909\,\text{in}^2.$$

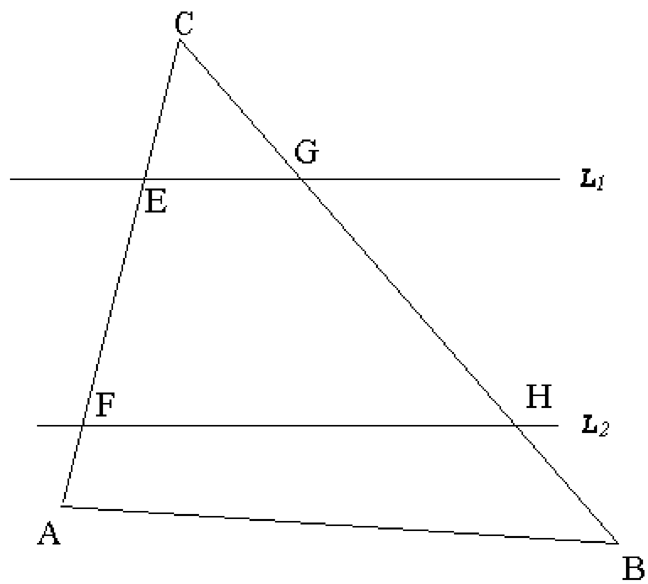
Exercises:

If $\measuredangle 1 = 85°19'56''$, what is the measure of $\measuredangle 2$?



1. $\measuredangle 2$ =____________

If line $\boldsymbol{L_1}$ is parallel to line $\boldsymbol{L_2}$ and $\measuredangle CAB = 76°59'$ and $\measuredangle CBA = 46°29'$ and $\measuredangle CEG = 61°29'$

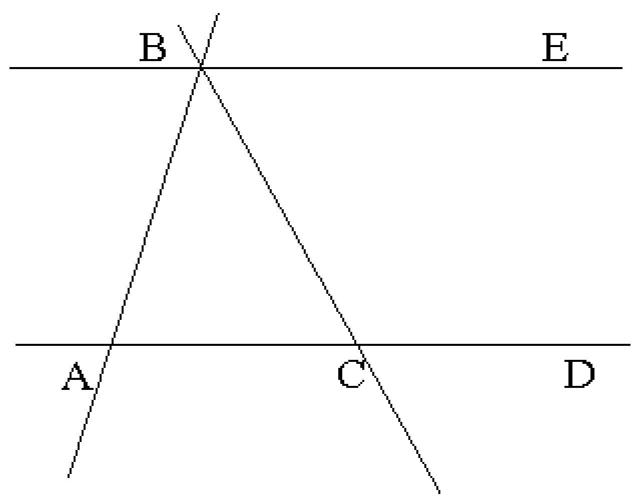


2. $\measuredangle$ACB = ____________

3. $\measuredangle$CFH = ____________

4. $\measuredangle$CGE = ____________

If **BE** is parallel to **AD** and $\measuredangle ACB = 64°17'$ and $\measuredangle BAC = 66°35'$, find the requested missing $\measuredangle$s.

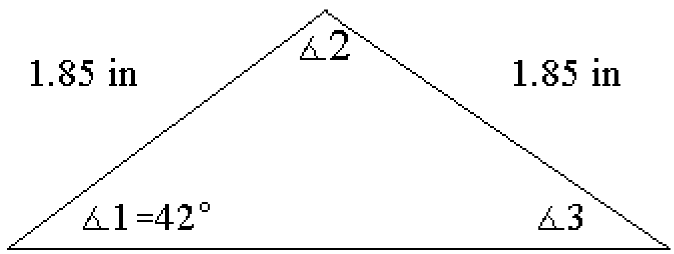


5. $\measuredangle$ABC = ____________

6. $\measuredangle$BCD = ____________

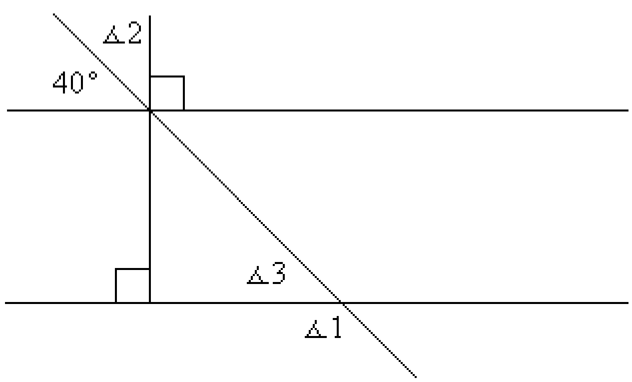
7. $\measuredangle$CBE = ____________

Find the measure of the missing angles.



8. ∠2 = ____________

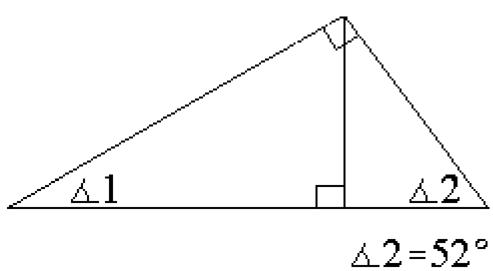
9. ∠3 = ____________



10. ∠1 = ____________

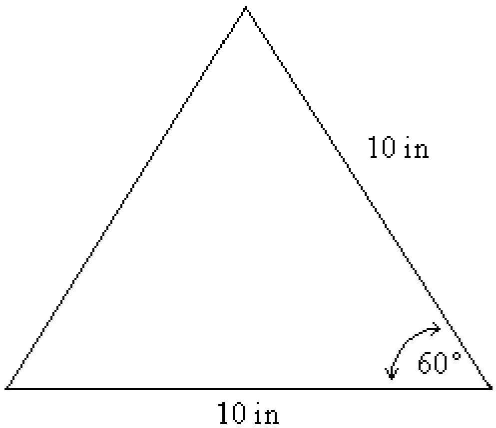
11. ∠2 = ____________

12. ∠3 = ____________

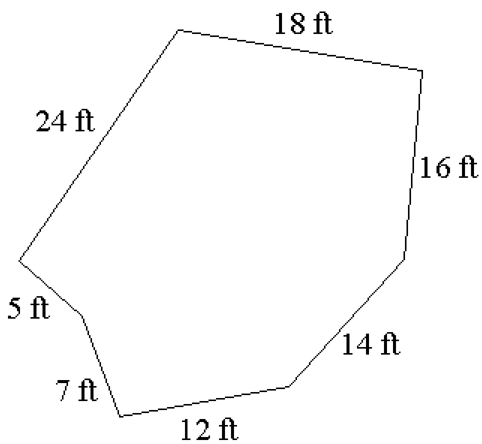


13. ∠1 = ____________

For each figure below calculate the distance around (perimeter) P.

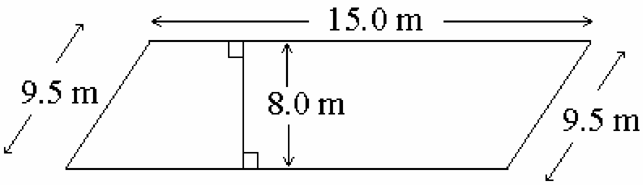


14. P = _____________



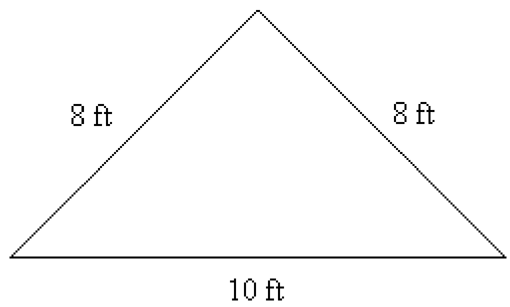
15. P = _____________

For each figure below calculate both the area, A, and the distance around (perimeter or circumference), P or C .



16. P =_____________

17. A =_____________

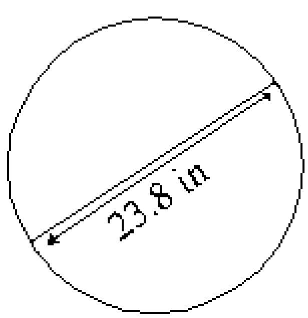


18. P =____________

19. A =____________

20. P =____________

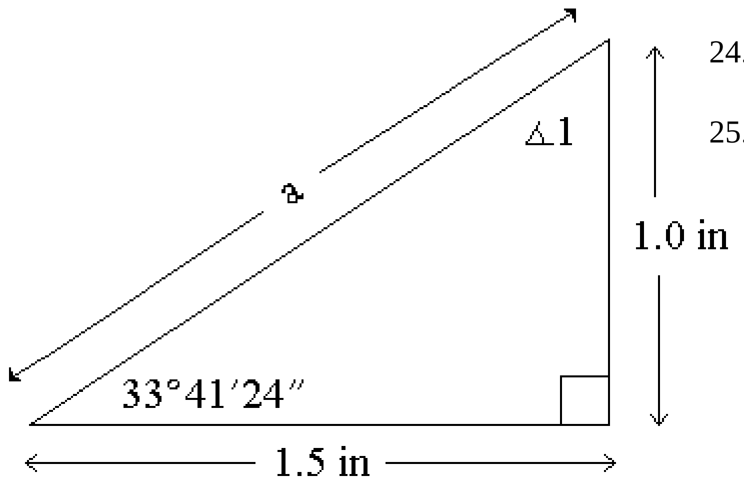
21. A =____________



22. C =____________

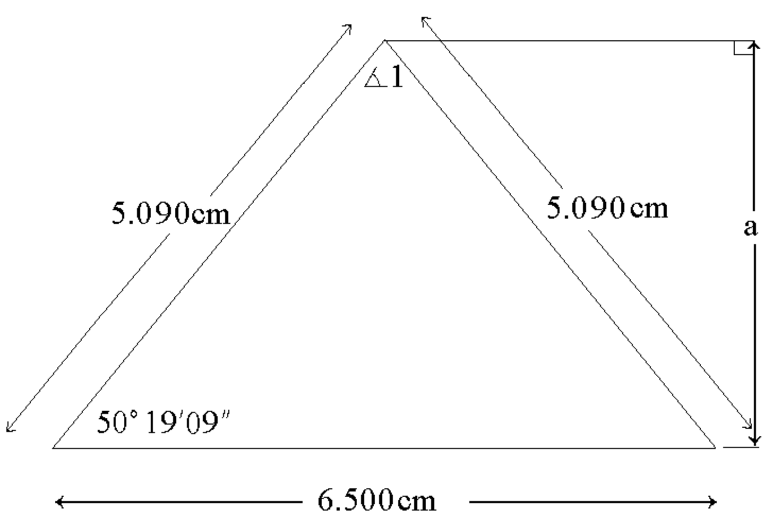
23. A =____________

For each figure below calculate the requested missing information.



24. ∡1 = ____________

25. a = ____________



26. ∡1 = ____________

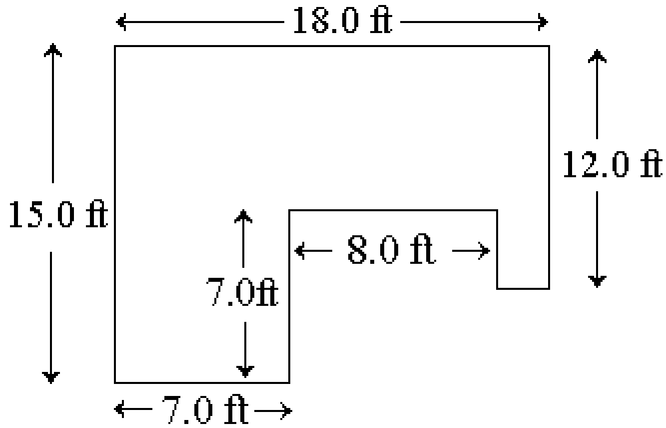
27. a = ____________

28. P = ____________

29. A = ____________

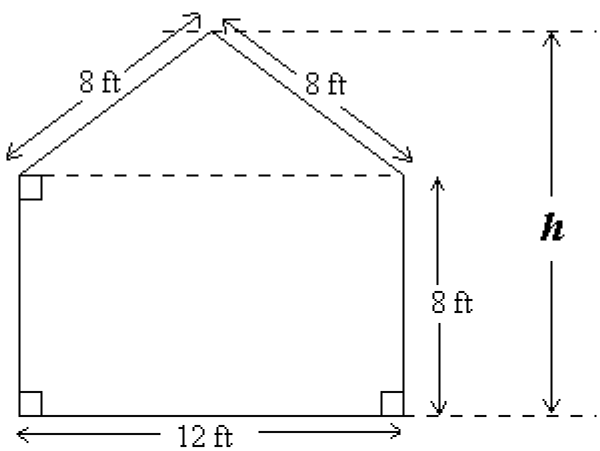
How many square feet of flooring does the following room have?

30) ______________



The cross section of a shed is shown below. What is the height *h* above the ground?

31. *h* = ______________



Determine A, the total area of the building's cross section.

32. A = ______________

A 2.25 cm diameter hole is drilled in a 4.5 cm diameter circle. What area is left after the hole has been drilled?

33) ______________

Answers :

1. $\angle 2 = 94°40'04''$; 2. $\angle ACB = 56°32'$; 3. $\angle CFH = \angle CEG = 61°29'$
4. $\angle CGE = 180° - \angle ACB - \angle CEG = 61°59'$; 5. $\angle ABC = 180° - \angle BAC - \angle ACB = 49°08'$
6. $\angle BCD = 180° - \angle ACB = 115°43'$; 7. $\angle CBE = \angle ACB = 64°17'$
8. $\angle 2 = 180° - 84° = 96°$; 9. $\angle 3 = 42°$; 10. $\angle 1 = 180° - 40° = 140°$; 11. $\angle 2 = 50°$
12. $\angle 3 = 40°$; 13. $\angle 1 = 38°$; 14. P = 30 in ; 15. P = 96 ft ; 16. P = 49.0 m ; 17. A = 120. m^2
18. P = 26 ft ; 19. A = 31.2 ft^2 ; 20. P = 58.5 in ; 21. A = 152. in^2 ; 22. C = 74.8 in
23. A = 445. in^2 ; 24. $\angle 1 = 56°18'36''$; 25. a = 1.80 in ; 26. $\angle 1 = 79°21'42''$
27. a = 3.92 cm ; 28. P = 16.7 cm ; 29. A = 12.7 cm^2 ; 30. 205 ft^2 ; 31. h = 13.3 ft
32. A = 128. ft^2 ; 33. 11.9 cm^2