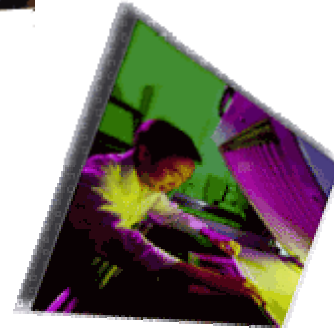


Introduction to College Mathematics 804-106



Text and Workbook

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3/12/09

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How to use this workbook

Each chapter consists of text plus worked examples. These are followed by **Exercises** labeled as **Your Turn**. You should work through each chapter checking the answers of the sample problems on your calculator. After this you should **work** the exercises. The answers are provided at the end of the chapter. Avoid looking at the solution before attempting the problem yourself. This often leads to a false sense of understanding. A good practice might be to do all the problems, referring to the unit examples as necessary, and then checking the answers with the text. Any problems missed should then be reviewed closely until you fully understand the solution. Working through the chapter sample exam will help you review all of the chapter's material. **Good Luck!**

Chapter 1 - Preliminaries

Section 1.1 Calculator Use

Throughout most of human history computation has been a tedious task that was often postponed or avoided entirely. It is only in the last generation that the use of inexpensive handheld calculators has transformed the ways that people deal with quantitative data. Today the use and understanding of electronic computation is nearly indispensable for anyone engaged in technical work. There are a variety of inexpensive calculators available for student use. Some even have graphing and/or symbolic capabilities. Most newer model calculators such as the Casio models *fx-300W*, *fx-300MS*, *fx-115MS*, and the Texas Instruments models TI-30X IIB, TI-30X IIS, TI-34 II enter calculations in standard “algebraic” format. Older calculators such as the Casio *fx-250HC* and the Texas Instruments TI-30Xa and TI-36X enter some calculations in a “reverse” format. Both types of calculators are priced under twenty dollars, yet possess enough computational power to handle the problems faced in most everyday applications. The Casio *fx-300MS* is fairly representative of the “newer” format calculators and the TI-30Xa is typical of “older” format ones. While other calculator models have similar or even better features for performing the required computations, the reader will be responsible for learning their detailed use. **Never** throw away the user’s manual!

In order to perform a computation, the correct keystrokes must be entered. Although calculators differ in the way keystrokes are entered, this text attempts to provide the reader with a couple of different keystroke options for each example problem in this chapter. The reader should practice the order in which to press the keys on the calculator while reading through the examples. This practice will ensure that the reader knows how to use his/her particular calculator. In order to indicate the sequence of keystrokes the following notation will be used. Digits (0 through 9 plus any decimal point) will be presented in normal typeface. Any additional keystrokes will be enclosed in boxes. For example, to multiply 7 times 8, the command sequence will be written as $7 \boxed{\times} 8 \boxed{=}$ and 56 appears on the display.

Section 1.2 Order of Operations

Mathematical expressions which involve more than one operation appear ambiguous. For example, is

$$5 + 6 \times 2 = 11 \times 2 = 22 \quad \text{or} \quad 5 + 6 \times 2 = 5 + 12 = 17 \quad ?$$

To clarify this question, mathematics has developed the following hierarchy of computations called **order of operations**.

1. Perform all operations that appear in grouping symbols first. If grouping symbols are nested, do the innermost first.
2. Raise all bases to powers in the order encountered moving from left to right.
3. Perform all multiplications/divisions in the order encountered moving from left to right.
4. Perform all additions/subtractions in the order encountered moving from left to right.

Here grouping symbols means parentheses (), brackets [], braces { }, etc. An example of a nested expression is $(6 + 2 \times (4 + 1)) \div 8$. The innermost grouping symbol is $(4 + 1)$ so the result is $(6 + 2 \times 5) \div 8 = (6 + 10) \div 8 = 16 \div 8 = 2$. Raising a base to a power (also known as an **exponent**) means repeated multiplication as in $6^3 = 6 \times 6 \times 6 = 216$. To perform this calculation on some calculators enter the following keystrokes: 6 $\boxed{\wedge}$ 3 $\boxed{=}$. The $\boxed{\wedge}$ key is called a carrot key. If your calculator does not have the $\boxed{\wedge}$ carrot key, it probably has the $\boxed{y^x}$ exponent key. To perform this same calculation on this type of calculator enter the following keystrokes: 6 $\boxed{y^x}$ 3 $\boxed{=}$. We will cover exponents in greater detail in **Section 1.4 - Decimal Fractions**.

In the original problem posed above the multiplication of 6 with 2 is performed before the addition of 5. The proper answer is therefore 17. The other interpretation could be achieved by using parentheses $(5 + 6) \times 2 = 11 \times 2 = 22$. Order of operations is built into all scientific calculators. That is, if you enter the keystrokes in the correct order, the calculator will automatically perform the correct calculation. Therefore 5 $\boxed{+}$ 6 $\boxed{\times}$ 2 $\boxed{=}$ yields the correct answer of 17. Try your own calculator to see if you get the correct answer. Then you will know if your calculator performs the order of operations correctly or not.

In many formulas x occurs as a variable, but then confusion with the times sign can result. To avoid this, alternative symbols for multiplication are used. They are the dot notation and adjacent parentheses as in $7 \times 3 = 7 \cdot 3 = (7)(3) = 21$. Some calculators **recognize** the adjacent parentheses as multiplication, but some do **not**. On these calculators the times operation $\boxed{\times}$ must be inserted between the parentheses.

Division is also indicated by a variety of notations. For example, the following all mean 34 divided by 17,

$$34 \div 17 = 34/17 = \frac{34}{17} = 17 \overline{)34} = 2 .$$

In addition to parentheses, brackets and braces, certain symbols act as **implied grouping symbols**. The most important of these are the fraction bar and the square root symbol.

The fraction bar acts to separate the numerator from the denominator. If either or both of the numerator or denominator consist of an expression with operations, these must be performed first before the division indicated by the fraction bar. For example,

$$\frac{7+3}{2+3} = \frac{10}{5} = 2 \quad . \quad \text{To perform this computation on the calculator, parentheses need to}$$

be inserted around both numerator and denominator as shown below.

$$\boxed{(} \boxed{7} \boxed{+} \boxed{3} \boxed{)} \boxed{\div} \boxed{(} \boxed{2} \boxed{+} \boxed{3} \boxed{)} \boxed{=}$$

Parentheses are the only grouping symbol the calculator recognizes or uses.

The square root symbol also acts as a grouping symbol. Any calculation inside the square root needs to be completed before the root is taken. For example, $\sqrt{25+144} = \sqrt{169} = 13$.

To perform this computation on the calculator parentheses need to be inserted around the expression inside the square root symbol.

On some calculators enter the following keystrokes : $\boxed{\sqrt{}} \boxed{(} \boxed{25} \boxed{+} \boxed{144} \boxed{)} \boxed{=}$.

On other calculators enter the keystrokes: $\boxed{(} \boxed{25} \boxed{+} \boxed{144} \boxed{)} \boxed{\sqrt{}} \boxed{=}$.

Note: On “newer” calculators like the Casio *fx-300MS* one enters expressions the way “they look” , i.e., the square root symbol comes first. On older models like the TI-30Xa

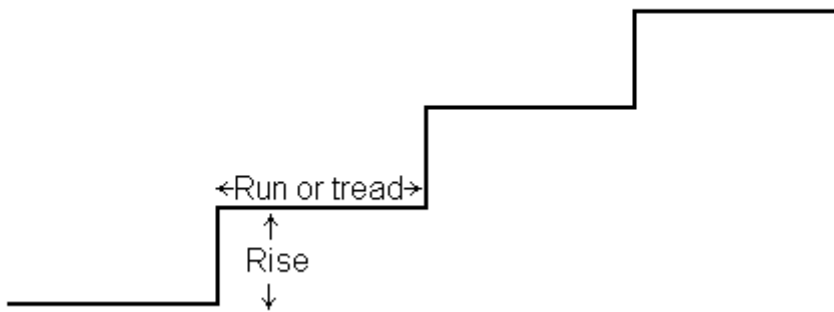
some functions like $\boxed{\sqrt{}}$ come after the expressions they are to evaluate. Give the above exercise a try to find out how your calculator works. In any case, using parentheses keys when necessary is a good habit to acquire. Failure to do so usually results in wrong answers!

Your Turn!!

Perform the following arithmetical operations. Remember to follow the correct order of operations.

$$1. \quad 12 \times 9 - 8 \times 7 \div 4 \times 2 \quad = \underline{\hspace{2cm}} \quad 2. \quad 12 \times 9 - 8 \times 7 \div (4 \times 2) \quad = \underline{\hspace{2cm}}$$

$$3. \quad 4(19 - 12) \div 2 - 11 \quad = \underline{\hspace{2cm}} \quad 4. \quad 32 - 3 \times 9 + 16 \div 4 \times 2 \quad = \underline{\hspace{2cm}}$$



A stairway consists of rises of 6 in and must reach a height of 10 feet. How many rises are needed?

19) _____

A stairway consists of 4 in rises and treads of 18 in. If the height of the stairs is 4 feet, what is the distance taken up by the stairway on the lower floor?

20) _____

Section 1.3 Fractions

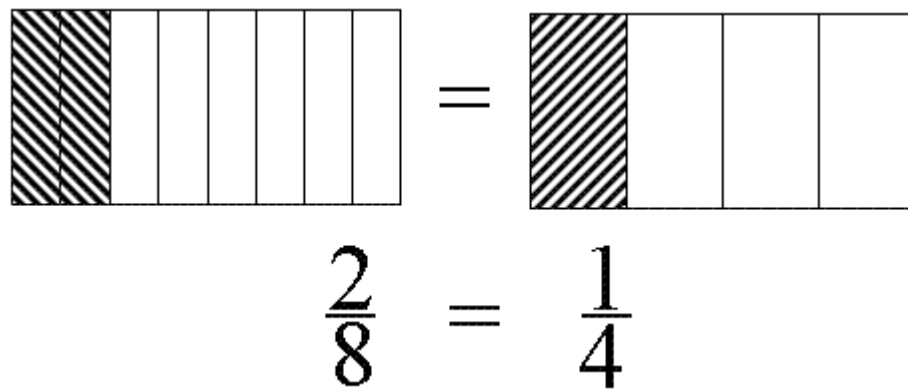
Fractions are ratios of whole numbers, which allow us to express numbers which are between the whole numbers. For example,

$$2\frac{2}{3} = 2 + \frac{2}{3} \text{ is between 2 and 3 .}$$

Fractions represent “part of a whole”. Imagine that we have a freight car with eight equal sized compartments. If three of these compartments are full of grain, we would indicate that we have three eighths of a freight car’s worth of grain. This is illustrated below.



Consider a car with eight compartments of which two are full. The fraction of a full car is two eighths. If we look at the same car split into four equal compartments, this same amount of grain fills one fourth of the car. We arrive at the following result.



We say that such equal fractions while they “look different” are **equivalent**. To generate equivalent fractions, we can multiply or divide both numerator (the top number) and denominator (the bottom number) by a common number. So we have the following fractions equivalent to two thirds.

$$\begin{aligned}\frac{2}{3} &= \frac{2 \times 3}{3 \times 3} = \frac{6}{9} \\ &= \frac{2 \times 17}{3 \times 17} = \frac{34}{51}\end{aligned}$$

Similarly, $\frac{18}{24} = \frac{18 \div 6}{24 \div 6} = \frac{3}{4}$. This same result could be stated in terms of “canceling” the common factor of 6 between the numerator and denominator.

$$\frac{18}{24} = \frac{6 \times 3}{6 \times 4} = \frac{\cancel{6} \times 3}{\cancel{6} \times 4} = \frac{3}{4}.$$

If a fraction has no common factors between its numerator and denominator, the fraction is in lowest terms.

There are three types of fractions.

1. Proper fractions with the numerator less than (symbolized by $<$) the denominator. All proper fractions are less than 1 .

2. Improper fractions with the numerator greater than (symbolized by $>$) the denominator. All improper fractions are greater than 1. Improper fractions can be expressed as a **mixed number**, which is a whole number plus a proper fraction. For example,

$$\frac{25}{6} = 6 \overline{)25} = 4 + \frac{1}{6} = 4\frac{1}{6} .$$

3. Unit fractions with the numerator equal to the denominator. All unit fractions are equal to one. For example,

$$\frac{19}{19} = \frac{4}{4} = \frac{25}{25} = \frac{1}{1} = 1 .$$

Note: when we write $4\frac{1}{6}$, we are using a shorthand notation. There really **is** a + sign between the 4 and the one sixth that's understood but unstated. Working backwards we can convert a mixed number into an improper fraction. For example,

$$7\frac{2}{3} = \frac{7 \times 3 + 2}{3} = \frac{23}{3} .$$

Fractions can be entered on many calculators using the $\boxed{\frac{b}{c}}$ key. For example, use the

following keystrokes to enter the fraction $\frac{14}{24}$: $14 \boxed{\frac{b}{c}} 24 \boxed{=}$.

$\boxed{7_12}$ will then appear in the display as the fraction reduced to lowest terms.

For a mixed number such as, $11\frac{5}{6}$ enter the following keystrokes:

$11 \boxed{\frac{b}{c}} 5 \boxed{\frac{b}{c}} 16 \boxed{=}$. Some calculators display $\boxed{11_5_6}$,

while other calculators display $\boxed{11_5_6}$. To change this answer to the improper

fraction $\frac{181}{16}$, enter $\boxed{\text{shift}} \boxed{\frac{b}{c}}$ on some calculators , or $\boxed{2nd} \boxed{\frac{b}{c}}$ on other calculators.

Also, on some calculators the $\boxed{\frac{b}{c}}$ key when pressed after entering a fraction converts it to a

decimal and if $\boxed{\frac{b}{c}}$ is pressed a second time the decimal is converted back to a fraction. On

other calculators fraction – decimal conversions are performed by entering $\boxed{2nd} \boxed{\leftarrow}$. The

$\boxed{\leftarrow}$ key is also the back space key, which deletes characters in the display.

As an application, solve for the following missing numerator: $\frac{\quad}{12} = \frac{6}{8}$. As a first step reduce $\frac{6}{8}$ to lowest terms as $\frac{3}{4}$. So $\frac{\quad}{12} = \frac{3}{4}$. The first denominator 12 is three times the second denominator 4, so the missing numerator must be three times the second numerator 3. The answer is that the missing numerator is 9.

To compare two fractions and determine which is larger, we can use the following procedure:

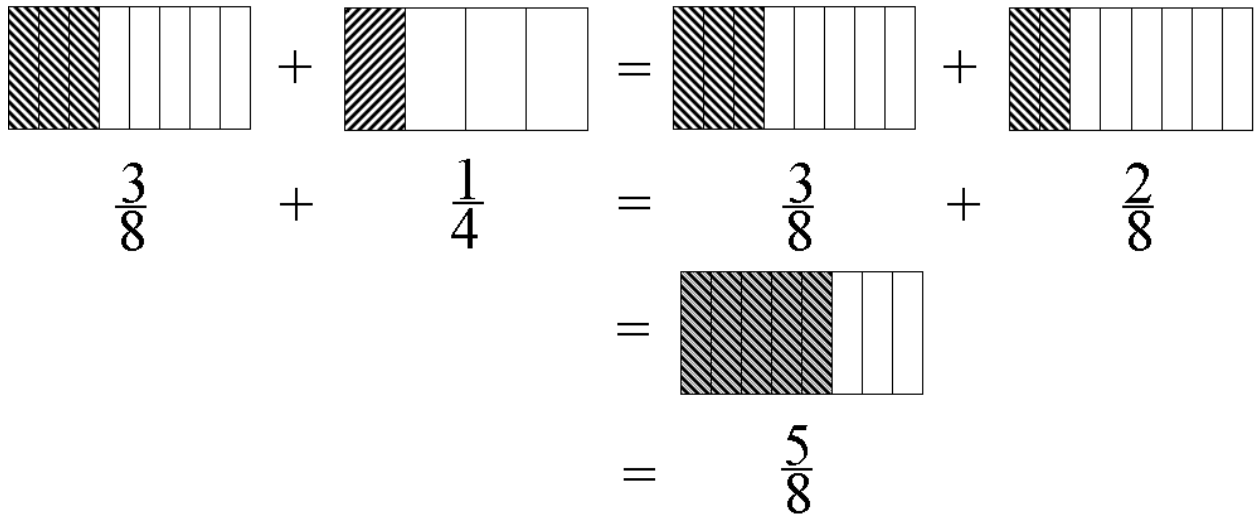
1. If the fractions involve mixed numbers with proper fractions, the number with the larger whole number is the larger number. For example,

$$7\frac{3}{16} > 5\frac{7}{8}, \text{ since } 7 > 5.$$

2. If the fractions are both proper fractions or mixed numbers with equal whole numbers, then convert the fractions into decimals. The number with the larger decimal is the larger number. For example,

$$5\frac{1}{3} < 5\frac{3}{8}, \text{ since } \frac{3}{8} = 0.375 > 0.333\dots = \frac{1}{3}.$$

To add or subtract fractions we need a common denominator. Consider adding $\frac{3}{8}$ to $\frac{1}{4}$. Since one fourth is equivalent to two eighths, we have the following solution:



If mixed numbers are involved, we first deal with the whole numbers, then the fractions. For example,

$$7\frac{1}{2} - 5\frac{9}{16} = 7 - 5 + \frac{1}{2} - \frac{9}{16} = 2 + \frac{8}{16} - \frac{9}{16} = 1 + \frac{16}{16} + \frac{8}{16} - \frac{9}{16} = 1 + \frac{16+8-9}{16} = 1\frac{15}{16}.$$

Note: Since $\frac{9}{16} > \frac{8}{16}$, we had to “borrow” $\frac{16}{16}$ from the 2.

To multiply fractions we form the product of the numerators over the product of the denominators. For example,

$$\frac{5}{8} \times \frac{3}{4} = \frac{5 \times 3}{8 \times 4} = \frac{15}{32} .$$

If the product involves mixed numbers, we first convert them to improper fractions. For example,

$$2\frac{5}{6} \times 4\frac{1}{5} = \frac{2 \times 6 + 5}{6} \times \frac{4 \times 5 + 1}{5} = \frac{17}{6} \times \frac{21}{5} = \frac{17}{3 \times 2} \times \frac{3 \times 7}{5} = \frac{17 \times 7}{2 \times 5} = \frac{119}{10} = 11\frac{9}{10} .$$

Note: we canceled the common factor of 3 between numerator and denominator in this calculation. In a multiplication problem this can always be done and saves the effort of later having to reduce the final answer. Also note that the answer is “reasonable” in that

$$2\frac{5}{6} \approx 3 \text{ and } 4\frac{1}{5} \approx 4 , \text{ so } 2\frac{5}{6} \times 4\frac{1}{5} \approx 3 \times 4 = 12 .$$

A quick estimation like this can often catch silly mistakes even when using a calculator!

Consider the division problem $8 \div 2 = 4$. This is the same as $\frac{8}{2} = \frac{8}{1} \times \frac{1}{2} = 8 \times \frac{1}{2} = 4$.

More generally, any division problem can be expressed as

$$a \div b = \frac{a}{b} = \frac{a}{1} \times \frac{1}{b} = a \times \frac{1}{b} .$$

This means that division by the number b is equivalent to multiplication by the fraction $\frac{1}{b}$. The fraction $\frac{1}{b}$ is called the **reciprocal** of $b = \frac{b}{1}$. To form the reciprocal of a number we exchange the numerator with the denominator. In summary, division by a **non-zero** number equals multiplication by the reciprocal of that number.

In a division problem 0 is never allowed as the denominator or divisor. The reason for this is as follows. Suppose $20 \div 0 = \frac{20}{0}$ made sense. Then there would be some number, a , which is the answer to this division problem. Restating this as a multiplication problem would give $a \times 0 = 20$. But any number times zero gives zero! So no sensible answer to $20 \div 0 = \frac{20}{0}$ exists.

Another way of explaining this goes to the very meaning of division. $20 \div 4 = \frac{20}{4} = 5$, means that 20 contains five 4's. How many 0's does 20 contain? There's no sensible answer to the question!

Consider now the division $\frac{1}{4} \div \frac{1}{8}$, from the diagram below it is clear that one fourth

contains 2 one eighths. So the answer must be 2 . The following shows that this result is consistent with the multiplication by the reciprocal definition of division.

$$\frac{1}{4} \div \frac{1}{8} = \frac{1}{4} \times \frac{8}{1} = \frac{8}{4} = 2 .$$

$$\frac{1}{4} \div \frac{1}{8} = \frac{1}{4} \times 8 = \frac{8}{4} = 2$$

If the division involves mixed numbers, we first convert them into improper fractions. For example,

$$4\frac{1}{2} \div 1\frac{1}{8} = \frac{9}{2} \div \frac{9}{8} = \frac{9}{2} \times \frac{8}{9} = \frac{8}{2} = 4 .$$

In expressions which combine operations the standard order of operations apply as shown in the following:

$$\frac{2}{3} \times 2\frac{1}{2} - 1\frac{1}{4} \div 3 = \frac{2}{3} \times \frac{5}{2} - \frac{5}{4} \times \frac{1}{3} = \frac{5}{3} - \frac{5}{12} = \frac{5 \times 4}{3 \times 4} - \frac{5}{12} = \frac{20-5}{12} = \frac{15}{12} = 1\frac{3}{12} = 1\frac{3 \times 1}{3 \times 4} = 1\frac{1}{4} .$$

These calculations are all easily performed on the calculator. The keystrokes for the previous calculation are as follows:

$$2 \left[\frac{b}{c} \right] 3 \left[\times \right] 2 \left[\frac{b}{c} \right] 1 \left[\frac{b}{c} \right] 2 \left[- \right] 1 \left[\frac{b}{c} \right] 1 \left[\frac{b}{c} \right] 4 \left[\div \right] 3 \left[= \right] .$$

More involved calculations with grouping symbols are also possible. For example,

$$\begin{aligned}
 3\frac{3}{4} - \left(5\frac{3}{16} - 3\frac{7}{8}\right) \div 2\frac{1}{2} &= 3\frac{3}{4} - \left(2 + \frac{3}{16} - \frac{14}{16}\right) \div \frac{5}{2} = 3\frac{3}{4} - \left(1 + \frac{16+3-14}{16}\right) \times \frac{2}{5} = \\
 &= 3\frac{3}{4} - 1\frac{5}{16} \times \frac{2}{5} = 3\frac{3}{4} - \frac{21}{16} \times \frac{2}{5} = 3\frac{3}{4} - \frac{21}{40} = 3 + \frac{30-21}{40} \\
 &= 3\frac{9}{40} .
 \end{aligned}$$

This is keystroked as follows:

$$3 \left[\frac{a}{c} \right] 3 \left[\frac{a}{c} \right] 4 \left[- \right] \left[(\right] 5 \left[\frac{a}{c} \right] 3 \left[\frac{a}{c} \right] 16 \left[- \right] 3 \left[\frac{a}{c} \right] 7 \left[\frac{a}{c} \right] 8 \left[) \right] \left[\div \right] 2 \left[\frac{a}{c} \right] 1 \left[\frac{a}{c} \right] 2 \left[= \right] .$$

Your Turn!!

Write as an improper fraction.

$$6\frac{11}{16}$$

1) _____

$$3\frac{7}{10}$$

2) _____

Write as a mixed number reduced to lowest terms.

$$\frac{19}{8}$$

3) _____

$$\frac{26}{10}$$

4) _____

Reduce to lowest terms.

$$\frac{9}{12}$$

5) _____

$$7\frac{18}{32}$$

6) _____

Supply the missing numerators:

$$\frac{3}{4} = \frac{?}{12}$$

7) _____

$$3\frac{3}{4} = \frac{?}{8}$$

8) _____

Indicate which number is larger.

$$\frac{19}{32}$$

$$\frac{5}{8}$$

9) _____

$$3\frac{7}{16}$$

$$2\frac{3}{4}$$

10) _____

Perform the indicated operations and express the answer as a fraction in lowest terms:

$$\frac{3}{16} \times \frac{4}{9}$$

11) _____

$$6 \div \frac{2}{3}$$

12) _____

$$\frac{3}{8} + \frac{5}{8}$$

13) _____

$$\frac{13}{32} - \frac{9}{32}$$

14) _____

$$4\frac{1}{8} \times 2\frac{1}{4}$$

15) _____

$$\frac{5}{8} \div \frac{15}{16}$$

16) _____

$$5\frac{3}{8} + 3\frac{11}{32}$$

17) _____

$$5\frac{3}{8} \div 2\frac{3}{4}$$

18) _____

$$2\frac{1}{4} \times 3\frac{2}{3}$$

19) _____

$$7\frac{19}{32} - 5\frac{3}{4} - 1\frac{5}{8}$$

20) _____

$$3\frac{1}{2} \times 3\frac{1}{4} - 10\frac{5}{8}$$

21) _____

$$1\frac{3}{8} - 4\frac{3}{4} \div 4$$

22) _____

Solve and state all results as fractions reduced to lowest terms.

How many pieces of $\frac{5}{16}$ inch thick plywood are in a stack 35 inches high?

23) _____

A lumberyard sells lumber only in even foot lengths. What is the shortest single board of lumber from which a carpenter could cut three $3\frac{1}{4}$ ft long and two $2\frac{3}{4}$ ft long pieces?

24) _____

A cubic foot contains about $7\frac{1}{2}$ gallons. How many cubic feet are there in 120 gallons?

25) _____

A nail $3\frac{1}{2}$ inches long, goes through a board $2\frac{3}{8}$ inches thick supporting a joist. How far into the joist does the nail extend?

26) _____

A part is measured as $2\frac{7}{16}$ inches long on a scale drawing. If the scale is one foot to $\frac{1}{2}$ inch, how long is the actual part?

27) _____

Section 1.4 Decimal Fractions

The difficulty in adding or subtracting fractions “by hand” compared to adding or subtracting whole numbers is obvious to anyone who has done such calculations. This difficulty motivated the development of representing fractions as decimal numbers. Using decimal fractions all arithmetical operations are similar to computations with whole numbers. The only complication is keeping track of the position of the decimal point.

The basis of the decimal representation of numbers is the use of **place value**. This allows us to represent an infinite range of numbers with only ten symbols (the digits 0 through 9). Contrast this with Roman Numerals or other early number systems where new symbols are constantly added to represent larger values. Place value uses the powers of 10.

$$\begin{aligned}
 10^0 &= 1 && \text{(The **definition** of an exponent of 0)} \\
 10^1 &= 10 \\
 10^2 &= 100 \\
 10^3 &= 1,000 \\
 10^4 &= 10,000 \\
 10^5 &= 100,000 \\
 10^6 &= 1,000,000 \\
 &\text{etc.}
 \end{aligned}$$

Note: 10^n is equal to 1 followed by n zeros.

When we write a number such as 27,483, the digit 2 stands not for 2, but for $2(10^4) = 20,000$. The digit 7 represents $7(10^3) = 7000$, etc. The value we associate with each digit comes from its place in the number. The right most digit of a whole number is in the “one’s place”, the second digit from the right is the “ten’s place”, etc. To extend the decimal system to fractions, we use the reciprocal powers of 10 and the decimal point to separate the “one’s place” from the “tenth’s place”.

$$10^{-1} = \frac{1}{10^1} = 0.1$$

$$10^{-2} = \frac{1}{10^2} = 0.01$$

$$10^{-3} = \frac{1}{10^3} = 0.001$$

$$10^{-4} = \frac{1}{10^4} = 0.0001$$

$$10^{-5} = \frac{1}{10^5} = 0.00001$$

$$10^{-6} = \frac{1}{10^6} = 0.000001$$

In general $10^{-n} = \frac{1}{10^n}$ is a decimal point followed by $n-1$ zeros.

The leading 0 to left of the decimal point is not required for a number smaller than 1. It is used to emphasize the location of the decimal point. A decimal fraction such as 0.375 is interpreted as

$$\begin{aligned} 0.375 &= 3 \times 10^{-1} + 7 \times 10^{-2} + 5 \times 10^{-3} \\ &= \frac{3}{10} + \frac{7}{100} + \frac{5}{1000} = \frac{375}{1000} = \frac{3 \times 125}{8 \times 125} = \frac{3}{8} \end{aligned}$$

Note: adding extra zeros to the right of the rightmost digit to the right of the decimal point does **not** change the value of the decimal fraction. It does, however, imply a greater knowledge of the precision of the value.

A decimal fraction like 0.375 is called a **terminating** decimal because the digits to the right of the decimal point come to an end. The procedure outlined above is how to convert a terminating decimal to a fraction. It is summarized below:

1. Carry along the digits to the left of the decimal point as the whole number part of the resulting mixed number. If there are no non-zero digits to the left of the decimal point, the decimal represents a proper fraction.
2. Put the digits to the right of the decimal point over the power of 10 that goes with the right most decimal place. For example, in converting 0.1145, 1145 is put over 10,000 since the right most digit, 5, is in the ten-thousandth's place.

$$0.1145 = \frac{1145}{10000} = \frac{5 \times 229}{5 \times 2000} = \frac{229}{2000}$$

3. Reduce this fraction to lowest terms.

To convert a fraction to a decimal is quite easy. We just translate the fraction bar into a division. Remember that in a mixed number there is an understood but unstated plus sign. So that

$$7\frac{11}{16} = 7 + 11 \div 16 = 7.6875 \dots$$

This can also be done directly using the calculator as was discussed in **Section 1.3 - Fractions**.

If a fraction is in lowest terms and its denominator has a factor besides 2 or 5, then that fraction, when converted to a decimal, will generate a **repeating** decimal. For example,

$$\frac{5}{12} = \frac{5}{2 \times 2 \times 3}, \text{ so } 12 \text{ has a factor of } 3, \text{ and } 5 \div 12 = 0.416666\dots = 0.41\bar{6}.$$

$$\frac{5}{12} = 0.416666\dots = 0.41\bar{6} = 0.416\bar{6} = 0.4166\bar{6}.$$

The 6's as indicated either by the ellipsis "... " or 6 with a bar on top repeat "forever".

Note: all of these ways of writing the repeating decimal are the same. Calculators will display 0.416666667 since they work with a fixed number of digits and will round the last digit displayed.

To convert a repeating decimal into a fraction is a little complicated and is rarely encountered in practical problems. As a result no problems requiring such a conversion occur in the unit exercises. However, if you are **curious**, the procedure is summarized and illustrated below:

1. Count and record the number of decimal places from the decimal point to the repeating string of digits.
2. Move the decimal point to the right by this number of places. The result is a decimal number where the repeating pattern of digits begins in the tenth's place immediately to the right of the decimal point.
3. The digits to the left of the decimal point of the result from Step 2 become the whole number part of a mixed number. If there are no non-zero digits to the left of the decimal point, then the original decimal began the repeating pattern with the first digit and the whole number part of the mixed number is zero.
4. Add the whole number from Step 3 to a fraction with the repeating digits as the numerator and a string of 9's as the denominator. The number of 9's in the string is equal to the number of repeating digits in the numerator.

5. Take the fraction from Step 4 and divide it by 10 raised to the power of the number from Step 1. This number, worked out as a fraction, is the fraction equivalent to the original repeating decimal.

To illustrate the steps convert $0.00\overline{666}\dots$ to a fraction.

Step 1. The number of places from the decimal point to the repeating string of 6's is two.

Step 2. The result is the decimal $0.666\dots$.

Step 3. The whole number is 0.

Step 4. There is one repeating digit, a 6, so the result is $0 + \frac{6}{9} = \frac{2}{3}$.

Step 5. Dividing two thirds by $10^2 = 100$ gives

$$\frac{2}{3} \div 10^2 = \frac{2}{3} \div 100 = \frac{2}{3} \times \frac{1}{100} = \frac{2 \times 1}{2 \times 50 \times 3} = \frac{1}{150}. \text{ So } 0.00\overline{6} = \frac{1}{150}.$$

As a more complicated example consider converting $3.1527272727\dots$ to a fraction.

Step 1. The number of places from the decimal point to the repeating string of 27's is two.

Step 2. The result is the decimal $315.272727\dots$.

Step 3. The whole number is 315.

Step 4. There are two repeating digits, 27, so the result is $315 + \frac{27}{99} = 315 + \frac{9 \times 3}{9 \times 11} = 315 \frac{3}{11}$.

Step 5. Dividing the answer of Step 4 by $10^2 = 100$ gives

$$315 \frac{3}{11} \div 100 = \frac{3468}{11} \times \frac{1}{100} = \frac{4 \times 867}{11} \times \frac{1}{4 \times 25} = \frac{867}{275} = 3 \frac{42}{275}.$$

Using a calculator we can verify that $3 \frac{42}{275} = 3 + 42 \div 275 = 3.15272727\dots = 3.15\overline{27}$.

Often we wish to approximate a decimal number by finding another decimal roughly equal to the first number, but expressed with less digits. This process is called **rounding**. To round use the following procedure:

- Determine the decimal place to which the number is to be rounded. Often this is stated in the problem or application.
- If the digit to the right of this decimal place is less than 5, then replace all digits to the right of this decimal place by zeros or discard them if they are to the right of the decimal point.
- If the digit to the right of the decimal place is 5 or greater, then increase the digit in this decimal place by 1 and replace all digits to the right of this decimal place by zeros or discard them if they are to the right of the decimal point.

As an example, consider rounding 10,547.395 to the different decimal places shown in the following table.

10,547.395 rounded to	Decimal Place of Rounding	Result
2 places	hundredth's place	10,547.40
1 place	tenth's place	10,547.4
the nearest unit	one's place	10,547.
the nearest ten	ten's place	10,550
the nearest hundred	hundred's place	10,500
the nearest thousand	thousand's place	11,000

Raising numbers to powers or exponents occurs in many applications. Recall that b^n means a product of n factors of b . The number b is called the **base**, and n is the **power** or **exponent**.

$$\text{So } 1.574^5 = 1.574 \times 1.574 \times 1.574 \times 1.574 \times 1.574 = 9.661034658.$$

This result is correct to as many places as your calculator will display. To perform this calculation on some calculators use the keystrokes $1.574 \text{ [^] } 5 \text{ [=]}$, while other calculators enter $1.574 \text{ [y}^x \text{] } 5 \text{ [=]}$.

Newer and/or graphing calculators generally use the “carrot” symbol \wedge for exponents. Exponents of two and three are very common and have special names; b^2 is called “**b squared**” and b^3 is called “**b cubed**”. Many calculators have $[x^2]$ keys to square a number. When evaluating an expression, the standard order of operations requires that bases be raised to powers before any multiplications or divisions are performed. This hierarchy is built into scientific calculators.

For example, consider evaluating $3.54 \times 7.21^3 - (10.7 \times 6.28)^2 \div 3.56$.

On some calculators this is done with the following keystrokes :

$$3.54 \text{ [x] } 7.21 \text{ [^] } 3 \text{ [-] } (\text{ [(] } 10.7 \text{ [x] } 6.28 \text{ [)] } \text{ [x}^2 \text{] } \text{ [÷] } 3.56 \text{ [=]}.$$

The display shows the answer as 58.46760266. The keystrokes on other calculators are identical except that the $[y^x]$ key is used instead of the $[^]$ key. Some calculators have an $[x^3]$ key, and this key could have been used instead of $[^] 3$ above.

Consider evaluating 25^{12} . Entering $25 \text{ [^] } 12 \text{ [=]}$ on the some calculators gives the display $5.960464477 \times 10^{16}$. Entering $25 \text{ [y}^x \text{] } 12 \text{ [=]}$ on other calculators results in $5.960464478 \times 10^{16}$.

Because of the large size of the number both calculators have expressed the result in **scientific notation**. In scientific notation we express the answer as a decimal number between 1 and 10

times ten to a power. Here the number between 1 and 10 is 5.960464478 and the power on 10 is 16. In ordinary decimal notation, which the calculator can't display for lack of space, this answer would be written as 59,604,644,780,000,000 . If you try to work with these large decimal numbers, the advantages of scientific notation soon become obvious!

Note: both calculators seem to suggest that the exponent applies to 5.960464478 . This is **not true**. The exponent is on ten, but to save space in the display the calculator does not show the 10.

Now consider $(0.04)^{12}$. Most calculators display $1.6777216 \times 10^{-17}$. The result is in scientific notation with a negative exponent on 10. In ordinary decimal notation this result would be 0.000000000000000016777216 . The left-most non-zero digit, 1, is 16 (17–1) decimal places to the right of the decimal point. Thus, in scientific notation a positive exponent on 10 gives the number of decimal places the decimal point must move to the right to get the ordinary decimal answer, while a negative exponent on 10 gives the number of decimal places the decimal point must move to the left to get the ordinary decimal answer.

To enter a number in scientific notation on some calculators, use the $\boxed{\text{EXP}}$ key. For example, to enter 6×10^{23} use the following keystrokes : 6.02 $\boxed{\text{EXP}}$ 23 . A very small number like 7.15×10^{-12} is entered with 7.15 $\boxed{\text{EXP}}$ $\boxed{(-)}$ 12 . Here $\boxed{(-)}$ is the “change sign” or minus key.

The procedure used on some other calculators is identical except that the $\boxed{\text{EE}}$ key is used instead of the $\boxed{\text{EXP}}$ key and the change sign key is $\boxed{+ \leftrightarrow -}$. Scientific notation will be covered more thoroughly in **Section 2.5**.

Consider a table of squares of the whole numbers.

N	N^2
0	0
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64
9	81
10	100
11	121
12	144

If we reverse this table, i.e., start with N^2 and get the value of N , the table would look like.

N^2	N
0	0
1	1
2	1.41421356
3	1.732050808
4	2
5	2.236067977
6	2.449489743
7	2.645751311
8	2.828427125
9	3
10	3.16227766
11	3.31662479
12	3.464101615

The second number is called the square root of the first. In symbols $N = \sqrt{N^2}$, for example, $3 = \sqrt{9}$. Remember that the square root symbol acts as a grouping symbol. Any operations inside the square root need to be completed before the root is taken. For example, $\sqrt{116-16} = \sqrt{100} = 10$.

To perform this computation on the calculator parentheses need to be inserted around the expression inside the square root symbol. On the some calculators enter

$\sqrt{\quad} \left(\quad 116 \quad - \quad 16 \quad \right) =$, while $\left(\quad 116 \quad - \quad 16 \quad \right) \sqrt{\quad} =$ are the corresponding keystrokes on other calculators.

A table of the cubes of whole numbers can also be formed.

N	N^3
0	0
1	1
2	8
3	27
4	64
5	125

If we reverse this table, i.e., start with N^3 and get the value of N , the table would look like.

N^3	N
0	0
1	1
2	1.25992105
3	1.44224957
4	1.587401052
5	1.709975947
6	1.817120593
7	1.912931183
8	2

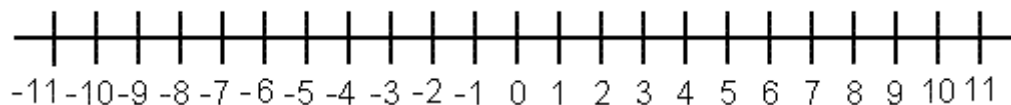
The second number is called the cube root of the first. In symbols $N = \sqrt[3]{N^3}$, for example, $3 = \sqrt[3]{27}$. The cube root, like the square root, acts as a grouping symbol. Any operations inside the cube root need to be completed before the root is taken. For example, $\sqrt[3]{85 \times 2 - 45} = \sqrt[3]{170 - 45} = \sqrt[3]{125} = 5$.

To perform this computation on the calculator parentheses need to be inserted around the expression inside the cube root symbol. On some calculators you will enter

$\boxed{\text{SHIFT}} \boxed{\sqrt[3]{}} \boxed{(} \boxed{85} \boxed{\times} \boxed{2} \boxed{-} \boxed{45} \boxed{)} \boxed{=}$, while the keystrokes on other calculators are $\boxed{(} \boxed{85} \boxed{\times} \boxed{2} \boxed{-} \boxed{45} \boxed{)} \boxed{2nd} \boxed{0} \boxed{=}$.

Section 1.5 Signed Numbers

When we are using numbers to express the change in a quantity, such as the amount of money in a checking account or a running back's total yards, we soon find that the quantities under study don't always increase. Bank accounts sometimes decline and running backs can lose yards! To represent a change, which decreases, we use negative numbers, while positive numbers represent an increase. A convenient way to visualize positive and negative numbers is the number line shown below. Here the positive (or "ordinary") numbers are to the right of zero and the negative numbers are to the left of zero.



The opposite of 5 is -5 since $-5 + 5 = 0$. For example, if you lose \$5 then make \$5, you're back to zero. By the same argument the opposite of -5 is 5. If a running back gains 10 yards, then

loses 7, his net yardage is 3 . In symbols, $10 + (-7) = 3$. So adding a negative 7 is the same as subtracting a positive 7 . Also $10 + (-7) = (-7) + 10 = 3$. In general, $a + (-b) = a - b$, i.e., subtracting is the same as adding the opposite and visa versa .

Suppose a running back loses 3 yards every time he carried the ball. If he had four carries, his net yardage is $-3 + (-3) + (-3) + (-3) = -12$. [You may have noticed that we don't write $+ -3$, but rather $+ (-3)$, this is just to avoid the potential confusion of two adjacent operation symbols.] However, using the definition of whole number multiplication as repeated addition, we see that $4(-3) = -3 + (-3) + (-3) + (-3) = -12$. So a positive number times a negative number should result in a negative number. What about a negative times a negative? One of the fundamental rules of arithmetic is called the distributive property. It says that

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad .$$

For example,

$$5 \cdot (4 + 7) = 5 \cdot 4 + 5 \cdot 7 = 20 + 35$$

$$5 \cdot 11 = 55 \quad .$$

Now,

$$(-1) \cdot (1 + (-1)) = (-1) \cdot 1 + (-1) \cdot (-1)$$

$$(-1) \cdot (0) = -1 + (-1) \cdot (-1)$$

$$0 = -1 + (-1) \cdot (-1) \quad .$$

So $(-1)(-1)$ added to -1 gives zero. But only 1 added to -1 makes zero. So we conclude that

$$(-1)(-1) = 1 \quad .$$

In general, a negative number times a negative number gives a positive number. We have analogous statements in English. If I say "I am not dishonest", the double negative makes the sentence equivalent to saying "I am honest".

We now have another way of forming the opposite of any number, simply multiply by -1 , i.e., $-b = (-1)b$. The standard order of operations requires that we square before multiplication. This means that $-5^2 = -1(5^2) = -25$, while $(-5)^2 = (-5)(-5) = 25$. Consider now subtracting a negative number as in $10 - (-8) = 10 + (-1)(-8) = 10 + 8 = 18$. So subtracting a negative is the same as adding the positive.

Finally, division is the opposite operation to multiplication. Since $(-5)(6) = -30$ and $(-5)(-6) = 30$, then

$$(-30) \div (-5) = 6$$

$$(-30) \div 6 = -5$$

$$30 \div (-5) = -6$$

$$30 \div (-6) = -5 \quad .$$

In general, a negative number divided by a negative number is a positive number, while a negative divided by a positive or a positive divided by a negative is negative.

To enter a negative number on some calculators use the minus sign key $\boxed{(-)}$

before the number just as it is written. For example, to evaluate

$$\frac{6 \times (-10)}{-3 - 2} = \frac{-60}{-5} = 12$$

enter the following keystrokes : $\boxed{(}$ $\boxed{6}$ $\boxed{\times}$ $\boxed{(-)}$ $\boxed{10}$ $\boxed{)}$ $\boxed{\div}$ $\boxed{(}$ $\boxed{(-)}$ $\boxed{3}$ $\boxed{-}$ $\boxed{2}$ $\boxed{)}$ $\boxed{=}$.

The keystrokes for some other calculators are shown below. **Note:** you enter the negative numbers “backwards”, i.e., first enter the value, then the change sign key.

$\boxed{(}$ $\boxed{6}$ $\boxed{\times}$ $\boxed{10}$ $\boxed{+ \leftrightarrow -}$ $\boxed{)}$ $\boxed{\div}$ $\boxed{(}$ $\boxed{3}$ $\boxed{+ \leftrightarrow -}$ $\boxed{-}$ $\boxed{2}$ $\boxed{)}$ $\boxed{=}$.

Your Turn!!

Perform the indicated operations giving answers to the stated number of decimal places:

7.11643 + 3.3489 (four places) 1) _____

27.32 – 6.972 (two places) 2) _____

0.25 × 0.4333 (two places) 3) _____

7.123 ÷ 1.48 (three places) 4) _____

\$57.23 – \$7.89 (two places, i.e., nearest penny) 5) _____

1.79² (two places) 6) _____

$\sqrt{0.144}$ (three places) 7) _____

2.53² × 1.96 – 5.36² ÷ 2.89 (two places) 8) _____

$$\sqrt{17^2 - 13^2}$$

(one place)

9) _____

$$\sqrt[3]{7 \times 8 - 11}$$

(two places)

10) _____

$$11.17^2 \times 5.10 - 5.97^3 \div 2.17$$

(one place)

11) _____

Write the following fractions as decimals:

$$2\frac{11}{16}$$

12) _____

$$1\frac{7}{12}$$

13) _____

Write the following decimals as a fraction in lowest terms:

0.45

14) _____

8.84

15) _____

Calculate the following:

$$-56 \div 8$$

16) _____

$$(-8)^2 \div (-16)$$

17) _____

$$-8^2 \div (-16)$$

18) _____

$$\frac{(-5)(-4)}{-10}$$

19) _____

$$\frac{(-2)^3(-3)}{-6}$$

20) _____

Solve the following problems:

A stack of eighteen pieces of lumber is 31.50 inches thick. How thick would a stack of thirty-three such sheets be?

21) _____

A delivery truck gets 11.3 miles per gallon of gasoline. If gas costs \$4.12 per gallon, what will be the cost of the gasoline needed to drive 189 miles? (Round to the nearest penny.)

22) _____

A machinist earns \$12.50 an hour plus time and a half for overtime (hours worked beyond 40). What is the machinist's gross pay for a 53.75 hour work week? (Round to the nearest penny.)

23) _____

In the first six months of the year, Precision Auto Body had the following profit and loss record:

January	\$8,736.52	profit
February	\$12,567.34	profit
March	\$1,282.72	loss
April	\$478.68	profit
May	\$179.66	loss
June	\$1,257.23	profit

Find the total profit or loss for this six month period.

24) _____

A welder earned \$468.75 (gross pay before deductions) for 37.5 hours of work. Find her hourly rate of pay.

25) _____

A cubic foot holds 7.481 gallons. A car has a gas tank which holds 14.5 gallons. To three decimal places, how many cubic feet is this?

26) _____

Chapter 1 Practice Exam**Perform the indicated operations.**

1. $2168 \times 15 - 18549$

2. $19 \times 13 - 70 \times 5 \div 10 \times 7$

3. $9 \times 13 - 70 \times 5 \div (10 \times 7)$

4.
$$\frac{7 \times 5 - 6 \times 4}{7 \times 4 - 3 \times 8}$$

5. Write as an improper fraction: $5\frac{7}{32}$

6. Reduce to lowest terms: $\frac{12}{81}$

7. Supply the missing numerator: $\frac{13}{32} = \frac{?}{96}$

8. Which number is larger: $\frac{17}{29}$ or $\frac{31}{45}$

Perform the indicated operation, express result in lowest terms:

9. $\frac{7}{9} \times 4\frac{3}{5}$

10. $6 \div \frac{5}{12}$

11. $\frac{3}{8} + \frac{5}{32}$

12. $7\frac{1}{8} - 5\frac{1}{4}$

13. $5\frac{1}{3} \div 3\frac{5}{6}$
14. $15\frac{5}{64} - 5\frac{7}{8} - 6\frac{3}{16}$
15. $6\frac{1}{8} + 3\frac{2}{3} \times 1\frac{3}{4}$
16. Perform the indicated operation, express result to two decimal places: $4.65 \times 3.29 + 5.46$
17. Perform the indicated operation, express result to three decimal places: 2.176^2
18. Perform the indicated operation, express result to one decimal places: $\sqrt{15^2 + 12^2}$
19. Perform the indicated operation, express result to two decimal places:
 $\sqrt{3.14159 \times 5.67^2 - 40.34}$
20. Write $7\frac{9}{16}$ as a decimal.
21. Write 0.625 as a fraction in lowest terms.
22. Calculate $(10 + 2)^2 \div (20 - 5)^2 \times (3^2 - 7)$
23. A steam fitter earns \$18.60 per hour plus time and a half for overtime (more than 40 hours in one week). If she works $65\frac{1}{2}$ in one week, what is her gross pay for that week?
24. A car averages 61.7 miles per hour on a trip that takes $5\frac{3}{4}$ hours. What is the distance that the car traveled on the trip?
25. A wall's length is measured on a scale drawing as $6\frac{5}{8}$ inches. If the scale is a half-inch on the drawing represents one foot in reality, what is the actual length of the wall?

Chapter 1 Answer Section

Section 1.2 Solution Set:

1. 80 ; 2. 101 ; 3. 3 ; 4. 13 ; 5. 11 ; 6. 13 ; 7. 5 ; 8. 16 ; 9. 5 ; 10. 68 ; 11. 189 ;
 12. 14.5 ; 13. 91 inches ; 14. \$25,100,900,000; 15. \$45 ; 16. \$938 ; 17. \$192 ;
 18. 400 miles ; 19. 20 ; 20. 216 in or 18 ft

Section 1.3 Solution Set:

1. $\frac{107}{16}$; 2. $\frac{37}{10}$; 3. $2\frac{3}{8}$; 4. $2\frac{3}{5}$; 5. $\frac{3}{4}$; 6. $7\frac{9}{16}$; 7. $\frac{9}{12}$, missing numerator is 9
 8. $\frac{30}{8}$, missing numerator is 30 ; 9. $\frac{19}{32} < \frac{5}{8}$; 10. $3\frac{7}{16} > 2\frac{3}{4}$; 11. $\frac{1}{12}$; 12. 9
 13. 1 ; 14. $\frac{1}{8}$; 15. $9\frac{9}{32}$; 16. $\frac{2}{3}$; 17. $8\frac{23}{32}$; 18. $1\frac{21}{22}$; 19. $8\frac{1}{4}$; 20. $\frac{7}{32}$
 21. $\frac{3}{4}$; 22. $\frac{3}{16}$; 23. 112 pieces ; 24. A board 16 ft long ; 25. 16 cubic feet
 26. $1\frac{1}{8}$ in ; 27. $4\frac{7}{8}$ ft

Section 1.5 Solution Set:

1. 10.4653 ; 2. 20.35 ; 3. 0.11 ; 4. 4.813 ; 5. \$49.34 ; 6. 3.20 ; 7. 0.379 ; 8. 2.60 ;
 9. 11.0 ; 10. 3.56 ; 11. 538.3 ; 12. 2.6875 ; 13. 1.58333... ; 14. $\frac{9}{20}$; 15. $8\frac{21}{25}$; 16. -7 ;
 17. -4 ; 18. 4 ; 19. -2 ; 20. -4 ; 21. 57.75 in = 4 ft $9\frac{3}{4}$ in ; 22. \$68.91 ; 23. \$757.81
 24. a profit of \$21,577.39 ; 25. \$12.50 ; 26. 1.938 cubic feet

Chapter 1 Practice Exam Solution Set:

1. 13971 ; 2. 2 ; 3. 112 ; 4. 2.75 or $\frac{11}{4}$ or $2\frac{3}{4}$; 5. $\frac{167}{32}$; 6. $\frac{4}{27}$; 7. 39 ; 8. $\frac{31}{45}$; 9. $\frac{161}{45}$ or $3\frac{26}{45}$;
 10. $\frac{72}{5}$ or $14\frac{2}{5}$; 11. $\frac{17}{32}$; 12. $1\frac{7}{8}$ or $\frac{15}{8}$; 13. $1\frac{9}{23}$ or $\frac{32}{23}$; 14. $3\frac{1}{64}$ or $\frac{193}{64}$; 15. $12\frac{13}{24}$ or $\frac{301}{24}$;
 16. 20.76 ; 17. 4.735 ; 18. 19.2 ; 19. 7.79 ; 20. 7.5625 ; 21. $\frac{5}{8}$; 22. 1.28 ; 23. \$1455.45 ;
 24. 354.8 miles ; 25. 13.25 ft or 13 ft 3 in or 159 in

Chapter 2 - Algebra

Algebra, despite its fearsome reputation, is really just "generalized" arithmetic. The name itself comes from the title of a ninth century Arabic mathematics text *Ilm al-jabr wa'l mugabalah* written by Al-Khowarozmi in the court of Al-Maman, the Caliph of Bagdad. Loosely translated, this is the art of working backwards to reach a solution. The modern word "algebra" has come to mean any of the mathematics involving the use of letters to symbolize an unknown number.

Basic vocabulary

Any letter used to represent a number value is called a **variable**. The letter x is the most commonly used variable, but a variable can be any letter of the alphabet, capital or lower case, or even other symbols. Variables are used in place of actual numbers when the true value is unknown or when the number may change. Algebra is very useful for analyzing situations where two or more quantities depend on one another.

An **operation** is any mathematical instruction between two numbers or variables that calls for a specific calculation. The basic mathematical operations are addition, subtraction, multiplication, and division.

An **expression** is a meaningful collection of numbers, variables, and operations. " $27 - 10$ " is a numeric expression because it has no variables, just a couple numbers and the subtraction operation. " $9a - 5b$ " is a variable expression, because it has variables (a and b) as well as numbers (9 and 5) with operations (multiplication and subtraction).

Three algebraic concepts: Evaluate, Simplify, and Solve

All of the work in algebra can be classified as one of three main tasks to perform.

Evaluate means to use a given or known value for the variable to find a single numerical value for the answer. We can replace variables with actual values when they are known. A common use of evaluation is to replace letters in a formula with specific numbers to determine an answer.

For example, the formula $C = \frac{5}{9}(F - 32)$ can be used to convert a Fahrenheit temperature F to an equivalent Celsius temperature C .

Simplify means to rewrite an expression with variables in a simpler, but equivalent form. It is preferable to work with fewer symbols and smaller numbers whenever it is possible to change an expression into a nicer form. For example, $4x + 3x$ can be simplified to $7x$, which is simpler because it does not have the addition symbol, and it has only one use of the variable x .

Solve means to use information (such as an equation) about a variable to determine its precise value (or all possible values, if more than one answer works). For example, the equation

$2x + 1 = 7$ is only true when $x = 3$. To solve the equation means to systematically find this value. The methods needed to reduce the original equation to the solution value make up the Algebra that Al-Khowarozmi was beginning to develop in his historic text.

Section 2.1 Evaluation

The first and most important step in evaluation is to follow the universal **order of operations**. This is a set of rules for making any series of mathematical calculations. When we have more than one operation in an expression, we must agree to follow the same order of operations, or we will never agree on the answers. All mathematics, worldwide, follows these same rules of order. Scientific and graphing calculators are also programmed to follow the correct order.

The order of operations can be summarized with the acronym “PEMDAS:”

- P) Work inside the **Parentheses** first. Parentheses are used to give priority to some part of the whole expression. Parentheses include grouping symbols of all shapes: () or [] or { }.
- E) Simplify all **Exponent** calculations next.
- MD) **Multiply** and **Divide**, working from left to right if there are more than one of these.
- AS) **Add** and **Subtract** any remaining terms, working left to right.

Evaluate a numeric expression

Follow the order of operations to simplify any expression involving numbers and operations to a single value.

Example Evaluate the expression $17 - 5 \cdot 2$.

Solution Multiplication is higher than subtraction in the order priority, so we must multiply $5 \cdot 2$ first and replace it with the result 10. The subtraction is done last.

$$17 - 5 \cdot 2 = 17 - 10 = 7$$

Note: The middle dot between two numbers indicates a multiplication, not a decimal point. The times symbol \times is avoided in algebra because it is too easy to confuse with the variable x . It is also customary to drop the symbol entirely, so whenever there is no symbol between two numbers, the operation is multiplication.

The expression $17 - 5 \cdot 2$ can also be written as $17 - 5(2)$. Here, there is no symbol between the number 5 and the number (2), so multiplication is assumed.

Example Evaluate the expression $3(5 + 4)$.

Solution In this case, the group in parentheses must be calculated first. Replace any expression inside parentheses with its answer before continuing with any operations outside the parentheses.

$$3(5 + 4) = 3(9) = 27$$

Example Evaluate the expression $5 + 11(8 - 5) - 16$.

Solution Again, the group in parentheses must be calculated first. Then do the multiplication, and finish with the addition and subtraction.

$$\begin{aligned} &5 + 11(8 - 5) - 16 \\ &= 5 + 11(3) - 16 \\ &= 5 + 33 - 16 \\ &= 38 - 16 \\ &= 22 \end{aligned}$$

Note: Addition and subtraction will always be the last operations performed, unless they are grouped inside parentheses.

Example Evaluate the expression $8 + (5 - 2)^2$.

Solution Again, the group in parentheses must be calculated first.

$$8 + (5 - 2)^2 = 8 + (3)^2$$

The raised “2” here is an **exponent**. Exponents are a short-hand for repeated multiplications. $(3)^2$ means to multiply two copies of the base number 3: $(3)(3) = 9$. Replace the exponent calculation $(3)^2$ with the result 9, and then continue following PEMDAS:

$$\begin{aligned} &8 + (5 - 2)^2 \\ &= 8 + (3)^2 \\ &= 8 + 9 \\ &= 17 \end{aligned}$$

Evaluate a variable expression

With a variable expression, individual variables may be assigned specific values. In this case, replace each letter with the known value, and then follow the order of operations to simplify the remaining numeric expression to a single value.

Example Evaluate the expression $a^2 - 2b$ when $a = 5$ and $b = 3$.

Solution Think of the variables as place holders. The best way to do this is to replace each letter with empty parentheses. This makes room for the actual value, which then fills in the parentheses. After filling in the given values for a and b , the exponent calculation has the highest priority, followed by the multiplication, and then the subtraction last.

$$\begin{aligned} & a^2 - 2b \\ & = (\quad)^2 - 2(\quad) \\ & = (5)^2 - 2(3) \\ & = 25 - 2(3) \\ & = 25 - 6 \\ & = 19 \end{aligned}$$

Example Evaluate the expression $y^2 - 2(x + 7)$ when $x = 4$ and $y = 9$.

Solution Replace each variable with the known value and then follow PEMDAS.

$$\begin{aligned} & y^2 - 2(x + 7) \\ & = (\quad)^2 - 2((\quad) + 7) \\ & = (9)^2 - 2(4 + 7) \\ & = 9^2 - 2(11) \\ & = 81 - 2(11) \\ & = 81 - 22 \\ & = 59 \end{aligned}$$

Your Turn!!

Evaluate the following numeric expressions:

1. $33 - 17$

2. $-(33 - 17)$

3. $7 + 3 - 6$

4. $4 \cdot 3 - 8$

5. $7 - 2(3)$

6. $5 + 3^2$

7. $8(5 - 3) - 3^2$

8. $4(9 - 6)$

9. $2 + 3(7 - 3)$

10. $5(2 + 3) - 3 \cdot 4^2$

11. $5^2 - 3(5) + 1$

12. $(-4)^2 - 3(-4) + 1$

13. $1.56 - 7.12 + 2.15^2$

14. $-7 - \frac{12}{3}(-4)^3$

15.
$$\frac{235.1 - 142.6(1.73)^2}{38.6 + 1.8^3(2.13)^2}$$

16. $-6^2 - (-2)^3$

(Round to two decimal places)

17. $(-6)^2 - (-2)^3$

18. $-(-6^2 - 2)^3$

Evaluate the following variable expressions with the given values:

19. $a^2 - b$ when $a = 2$ and $b = 3$

20. $4x^2$ when $x = 5$

21. $3a^2 + 7a - 12$ when $a = 2$

22. $2 + 3(w - x)^2$ when $w = 4$ and $x = 1$

23. $k^2 + 4k - 1$ when $k = -3$

24. $s^2t + 3r$ when $s = 5$, $t = 2$, and $r = 4$

25. $(x + 8y - xz)^2$ when $x = 7$, $y = 1$, and $z = 2$

26. $\frac{1}{2}bh$ when $b = 14$ and $h = 11$.

27. $\frac{5}{9}(F - 32)$ when $F = 77$

28. $\frac{h(a + b)}{2}$ when $a = 5.5$, $b = 2.5$, and $h = 6.5$.

Section 2.2 Simplifying Expressions

To simplify variable expressions, we need to learn what happens with different operations. Naturally, addition has a different effect than multiplication. In this section, we will look at some basic simplification techniques involving addition, subtraction, and multiplication in variable expressions.

Addition and subtraction of variable terms

Even if you have never worked with variables, you are probably already familiar with combining like terms. For example, the sum of 4 boxes and 5 boxes is 9 boxes. You simply add ($4 + 5 = 9$) and keep the “boxes” as the like units. Now let x be a variable that represents one box. The calculation becomes $4x + 5x = 9x$. In words, you could say “4 x ’s and 5 x ’s make 9 x ’s. This works for any variable: $7a - 20a = -13a$, and $-2b - 3b = -5b$.

The number in front of any variable is called the **coefficient** of the variable. The coefficient is the number attached a variable term by multiplication. To add or subtract variable terms, we combine the coefficients and keep the same variable (these are called **like terms**).

Note that we usually do not write a coefficient of 1, so x alone really means $1x$. Thus $9x + x = 10x$. This is consistent with our English language, where we might refer to “a box” without emphasizing that we specifically mean “one box.”

If two terms have different variable parts, then we cannot simplify the expression. For example, the expression $12b + 4w$ has no simpler form. How else could you write “12 *bananas* and 4 *watermelons*” without losing information? Also note that a constant term (one with no variable) is different from any variable term. If terms have different variables, leave them alone. If they have the same variable, you may combine the coefficients.

Examples $6y + 10y$ simplifies to $16y$

$3t - 21t$ simplifies to $-18t$

$62w - 70w$ simplifies to $-8w$

$-20 + 4x - 3$ simplifies to $4x - 23$

$8x - 3 + 7x + 9$ simplifies to $(8x + 7x) + (-3 + 9) = 15x + 6$

$9m + 3p - 4m + 12$ simplifies to $5m + 3p + 12$ (combine only the two m terms)

Variable terms must also have the same exponent to qualify as like terms. $3x^2$ and $5x^2$ are like terms and may be added: $3x^2 + 5x^2 = 8x^2$.

Note: The exponent is never changed by an addition or subtraction operation.

Example Simplify $30t^2 + 7t - 8 - 12t^2 + 3t + 5$.

Solution Group the like variable terms.

$$\begin{aligned} & 30t^2 + 7t - 8 - 12t^2 + 3t + 5 \\ &= (30t^2 - 12t^2) + (7t + 3t) + (-8 + 5) \\ &= 18t^2 + 10t - 3 \end{aligned}$$

Multiplication using the distributive property

The distributive property of real numbers shows how multiplication and addition/subtraction interact. For example, we can see that $4(2 + 3) = 4(5) = 20$ if we simply follow the order of operations. On the other hand, we can multiply first provided we multiply through to each term inside the parentheses: $4(2 + 3) = 4(2) + 4(3) = 8 + 12$, which is 20 again. This property always works, no matter what the numbers are.

In general, we can write the distributive property as $a(b + c) = ab + bc$ for all real numbers a , b , and c . In other words, when we have a multiplier in front of a parentheses group, we can choose to simplify and remove the parentheses provided that we multiply through to each term inside the parentheses. This is especially useful when variables are involved, because we may have no other way to simplify an expression with parentheses.

Example $2(x + 3) = 2(x) + 2(3) = 2x + 6$

Examples $5(p - 9) = 5(p) - 5(9) = 5p - 45$

$$3(4c - 2) = 3(4c) - 3(2) = 12c - 6$$

$$7(2x + 1) = 7(2x) + 7(1) = 14x + 7$$

With a negative coefficient in front of the parentheses group, all of the signs inside get changed.

Example Simplify $-2(f + 6)$.

Solution Multiply both terms inside the parentheses by -2 .

$$\begin{aligned} & -2(f + 6) \\ &= -2(f) - 2(6) \\ &= -2f - 12 \end{aligned}$$

Note: Multiplying two numbers with the same sign (both positive or both negative) always has a positive result. Multiplying two numbers with different signs (one positive and one negative) always has a negative result.

Example Simplify $-4(9v - 2)$.

Solution Multiply both terms inside the parentheses by -4 .

$$\begin{aligned} & -4(9v - 2) \\ &= -4(9v) - 4(-2) \\ &= -36v + 8 \end{aligned}$$

With more terms inside the parentheses, multiply the coefficient by every term:

Example Simplify $-6(2x - 5y - 10)$.

Solution Multiply all three terms inside the parentheses by -6 .

$$\begin{aligned} & -6(2x - 5y - 10) \\ &= -6(2x) - 6(-5y) - 6(-10) \\ &= -12x + 30y + 60 \end{aligned}$$

For expressions with two or more parentheses groups, simplify first by multiplying each group using the Distributive Property, and then combine any like terms from the two groups.

Example Simplify $2(4x + 9) + 3(2x + 1)$.

Solution Distribute the coefficient through each group first.

$$\begin{aligned} & 2(4x + 9) + 3(2x + 1) \\ &= 2(4x) + 2(9) + 3(2x) + 3(1) \\ &= 8x + 18 + 6x + 3 \\ &= 14x + 21 \end{aligned}$$

Example Simplify $2(m - 7) - 5(2m - 3)$.

Solution Note that the negative coefficient starting the second group changes signs on both of the following terms.

$$\begin{aligned} & 2(m - 7) - 5(2m - 3) \\ &= 2(m) + 2(-7) - 5(2m) - 5(-3) \\ &= 2m - 14 - 10m + 15 \\ &= -8m + 1 \end{aligned}$$

Your Turn!!

Simplify the following by combining like terms:

1. $11b - 2b$

2. $6h + 7h$

3. $4x + 9 - x$

4. $6x + 5 - 2x - 4$

5. $3y - 9x + 10 - y + 7x + 2$

6. $5w + 8t + 18 - 6t - 7 + 2w$

7. $x^2 + 6x + 15 + 3x^2 - 4x - 7$

8. $14 - 2y + 5y^2 - 10 - 4y - 2y^2$

Use the Distributive Property to simplify the following expressions:

9. $2(7x + 5)$

10. $6(x - 3)$

11. $4(3g - 2)$

12. $-2(4x + 3)$

13. $10(4a - 3b + c)$

14. $5(1 - x - 3x^2)$

15. $-(2x - 21)$

16. $-8(1 - 2b)$

17. $(9p - 8) + (3p + 2)$

18. $(6y + 5) - (4y - 11)$

19. $3(r + 8) + 5(2r - 3)$

20. $5(2x - 7) + 9(x - 3)$

21. $3(4f + 1) - 2(3f - 5)$

22. $2(x - 1) + 3(4x + 5) - 4(2x - 3)$

23. $(7y^2 - 2y + 1) - (3y^2 + 6y - 8)$

24. $4(3x^2 - 5x + 2) - 7(2x^2 - 6x - 3)$

Section 2.3 Solving Linear Equations

An **equation** is a mathematical statement involving two variable expressions and an equal sign ($=$). The equation makes the claim that there is some value of the variable that makes the two expressions equal at the same time. A value that makes the equation true is called a **solution** to the equation. To **solve** an equation means to find all values that make the equation true.

We will concentrate on techniques to solve a class of equations called **linear equations**. These are equations that have a variable but no exponents. All of the examples in this section are linear

equations. The following techniques all apply to more general equation types as well, but linear equations are the ones we will be able to solve completely without introducing more advanced techniques.

Addition property of equality

The addition property of equality can be used to solve simple equations that involve only the addition or subtraction operation. Although the solution to such equations may be fairly obvious, you should make an effort to write out the steps showing how you reach the solution. This will be good practice for solving more difficult equations later on. For more complicated equations, good work habits will make everything seem easier.

Example Solve $x - 5 = 4$

What is the solution to the equation $x - 5 = 4$? You likely said $x = 9$ without thinking too hard. How did you come up with 9? It is true that $9 - 5 = 4$ because $4 + 5 = 9$. Note that every subtraction problem has a related addition problem. We can solve the original equation by changing both sides of the equation in the same manner. In this case, we get the variable x alone by adding 5 to both sides. Since $-5 + 5 = 0$, the left side of the equation only has an x left:

Example Solve $x - 5 = 4$

Solution Add 5 to both sides to remove the -5 term

$$\begin{aligned} x - 5 &= 4 \\ +5 &= +5 \text{ (add 5 to both sides)} \\ x &= 9 \end{aligned}$$

We can always check our own answers after solving an equation. After doing the work to solve, use your answer in the original equation. If you get a true statement, then you know for sure that you have the right answer. If you get a false statement, then you must have made a mistake somewhere; go back and check your work. Using $x = 9$ in the original equation, we have $9 - 5 = 4$, which is true, so it must be the correct solution.

The **addition property of equality** guarantees that we can make changes (such as addition or subtraction) to an equation, provided that we treat both sides the same. If two expressions are equal, then we can add (or subtract) the same number on both sides, and the resulting expressions are still equal. We normally use the addition property of equality to create an equivalent, but simpler equation which is easier to solve than the original.

Example Solve $u + 19 = 7$

Solution Following the first example, we want to build an equivalent equation with the variable all alone on one side. To move the $+19$ term, we need to **subtract 19** from both sides.

$$\begin{aligned}
 u + 19 &= 7 \\
 \underline{-19} &= \underline{-19} \\
 u &= -12
 \end{aligned}$$

In some cases, we may need to simplify expressions in the equation first. There are two simple ways to simplify: (1) Multiply through any parentheses. (2) Combine like terms on each side of the equation.

Example Solve $7y - 2(3y + 4) = 9$

Solution We need to first multiply -2 through the group $(3y + 4)$. Then the two y -terms can be combined: $7y - 6y$ is just $1y$.

$$\begin{aligned}
 7y - 2(3y + 4) &= 9 \\
 7y - 6y - 8 &= 9 \\
 y - 8 &= 9
 \end{aligned}$$

After simplifying, now we can use the addition property to solve the equation.

$$\begin{aligned}
 y - 8 &= 9 \\
 \underline{+8} &= \underline{+8} \\
 y &= 17
 \end{aligned}$$

Multiplication property of equality

Above, we found that the addition property of equality preserves the solutions when we add or subtract the same number from both sides of an equation. We can also use a similar property to solve equations where the variable term has a coefficient (a multiplier). Since the opposite of multiplication is division, we can solve equations by dividing both sides by the same number. The **multiplication property of equality** guarantees that the solutions are preserved when we change an equation by multiplying or dividing both sides by the same number.

Example Solve equation $5c = 40$.

Solution Since $5 \times 8 = 40$, we see that c must be 8. Another way to look at this is $40 \div 5 = 8$. We can solve the equation by dividing both sides by 5.

$$\begin{aligned}
 \cancel{5}c &= \frac{40}{\cancel{5}} \\
 c &= 8
 \end{aligned}$$

In this case, $5c = 40$ shows the variable c multiplied by the number 5. The opposite of multiplying by 5 is dividing by 5. To keep the equation true, we must do the same thing to both sides.

Example Solve $3x = -30$.

Solution Here the x is multiplied by 3. The opposite operation is division by 3, so we divide both sides by 3.

$$\begin{aligned} \cancel{3}x &= \frac{-30}{\cancel{3}} \\ x &= -10 \end{aligned}$$

Check the answer in the original equation: -10 is the solution since $3(-10) = -30$.

Example Solve $1.4k = 4.06$

Solution Divide both sides by 1.4 to remove the multiplier.

$$\begin{aligned} 1.4k &= 4.06 \\ \frac{\cancel{1.4}k}{\cancel{1.4}} &= \frac{4.06}{1.4} \\ k &= 2.9 \end{aligned}$$

Check the answer in the original equation: $k = 2.9$ is the solution since $1.4(2.9) = 4.06$.

When the coefficient is negative, we can still solve by dividing off the coefficient, including the negative sign.

Example Solve $-9z = 47$

Solution Divide both sides by -9 .

$$\begin{aligned} -9z &= 47 \\ \frac{\cancel{-9}z}{\cancel{-9}} &= \frac{47}{-9} \\ z &= -\frac{47}{9} \end{aligned}$$

Check: $-9\left(-\frac{47}{9}\right) = +47$ (the 9's reduce out). Note that the answer may be left as an improper fraction. If the fraction does not reduce, leave the answer as is.

Solve equations that require simplification

Some equations may require simplification before you can divide. In particular, if an equation has two variable terms on the same side, combine the like terms together into a single variable term. Likewise, combine any number terms on the same side of the equation into a single number. Then we can proceed with solving the equation after the simplification is complete. Use the following procedure to step through the work needed to simplify and solve an equation.

Procedure for solving linear equations

- 1) First simplify each side of the equation as much as possible
 - a) Use the distributive property to remove parentheses groups
 - b) Combine like terms on each side.
- 2) Use the addition/subtraction properties of equality to move like terms to the same side of the equation. Separate the variable terms from the constant terms.
- 3) Use the multiplication property of equality to remove any remaining coefficient on the variable term by dividing both sides by the coefficient.
- 4) Check all solutions in the original equation to verify solutions. Correct for possible mistakes if your answers do not satisfy the equation.

Example Solve $3(2x + 5) = 4x + 9$

Solution First, distribute the multiplier to remove the parentheses.

$$6x + 15 = 4x + 9$$

Then collect like terms together on separate sides.

$$6x + 15 = 4x + 9$$

$$\underline{-4x} \quad = \underline{-4x}$$

$$2x + 15 = 9$$

$$\underline{-15} = \underline{-15}$$

$$2x = -6$$

Finish by dividing off the coefficient of the variable term.

$$\frac{2x}{2} = \frac{-6}{2}$$

$$x = -3$$

Example Solve $2(3h+4) - 9(h-2) = 4(5h+1)$

Solution First, distribute the multiplier to remove the parentheses and combine like terms on each side of the equation.

$$6h + 8 - 9h + 18 = 20h + 4$$

$$-3h + 26 = 20h + 4$$

Be very careful with the positive or negative signs on each term. The common one to miss is the double negative: $(-9)(-2) = +18$.

Then solve by moving like terms together on separate sides.

$$-3h + 26 = 20h + 4$$

$$\underline{+3h} \quad = \underline{+3h}$$

$$26 = 23h + 4$$

$$\underline{-4} = \quad \underline{-4}$$

$$22 = 23h$$

Remove the final multiplier using a division.

$$\frac{22}{23} = \frac{23h}{23}$$

$$\frac{22}{23} = h$$

It does not matter which side the variable ends up on, as long as it is isolated from all of the other numbers. This answer is equivalent to writing $h = \frac{22}{23}$. Some people prefer to move the variable term to the left side every time, but the equation in either order has the same meaning.

Your Turn!!

Solve the following equations.

1. $x + 5 = 9$

2. $x + 18 = 24$

3. $w + 7 = -19$

4. $y - 4 = -6$

5. $r - 10 = 12$

6. $x + 9 = -8$

7. $3y = 21$

8. $8x = 24$

9. $5f = 30$

10. $6c = 42$

11. $-3p = 21$

12. $4t = 14$

13. $-3 - 11 + k = -21 + 9$

14. $2w + 3w = 28 - 3$

15. $7x - 4x = 14 + 7$

16. $4z + 9 - 3z - 4 = 10$

17. $5x + 3 = 23$

18. $8x + 12 = -4$

19. $2k - 1 = 9$

20. $8 - 5x = -7$

21. $7w - 2 = 6w + 4$

22. $11x = 7x + 20$

23. $9y + 2 = 3y + 38$

24. $8x - 3 = 4x + 17$

25. $6x - 5 = 3x - 29$

26. $4r + 9 - 7 = 6r + 3r - 12$

27. $-7 - 4x - 5x = 9 - 6 - 5x$

28. $2p + 20 + 3 = 2p + 4p - 5$

29. $7(2x - 1) - 5x = x + 25$

30. $9(3q + 2) - 10q = 12q - 7$

31. $3v + 2(4v - 3) = 3(2v - 3)$

32. $7x + 3(2x + 5) = 5(2x + 3) + 2$

33. $9(y - 2) + 3(4y + 1) = 2(2y - 5)$

34. $3(5b - 2) - 2(4b - 5) = 2b + 7$

Section 2.4 Exponent Rules

Exponents are used so frequently in variable expressions that it will be useful to have rules for several common situations. Although every exponent expression can be simplified by

multiplying copies of the base, it is easier and quicker to apply the laws of exponents when appropriate. Each exponent law below is a consequence of the definition of powers: a^n represents the product of n copies of the base a . If you are ever unsure about which rule to apply, come back to this definition.

Simplify expressions of the form $a^m \cdot a^n$

This form applies any time we have a **product of two numbers with the same base**. Since the exponent counts the number of copies of the base, we need to find the total number of copies in the product. Note that $x^5 \cdot x^2$ means $(xxxxx)(xx) = xxxxxxx = x^7$. We can simplify the product of same bases by adding the exponents together. Normally, we do not write out the long string of variables (it is only done here to illustrate the idea). As a general rule, $a^m \cdot a^n = a^{m+n}$.

Examples $a^3 \cdot a^9 = a^{3+9} = a^{12}$

$$y \cdot y^4 = y^{1+4} = y^5$$

$$c^3(c^{10})(c^5) = c^{3+10+5} = c^{18}$$

$$(-5b^3)(7b^8) = -5 \cdot 7 \cdot b^3 \cdot b^8 = -35b^{11}$$

In short, we can multiply the coefficients and add the exponents to multiply on the same base.

Simplify expressions of the form $(ab)^n$

This rule covers any **power over a product**. Technically, the product inside the parentheses should be completed first, but this may not be possible when variables are involved. However, exponents are compatible with multiplication, so we can expand the power and collect the similar factors together. For example, $(ab)^3$ means $(ab)(ab)(ab)$, which can be regrouped as $aaabbb$ (the order of multiplication does not affect the product). In short, there are 3 a 's and 3 b 's in the product, so $(ab)^3 = a^3b^3$.

In general, the rule for a power over a product is $(ab)^n = a^n b^n$.

Examples $(xy)^5 = x^5 y^5$

$$(2pq)^4 = 2^4 p^4 q^4 = 16p^4 q^4$$

$$(-3t)^2 \text{ means } (-3t)(-3t) = +9t^2$$

Note: The parentheses make a big difference in the calculations because they change the normal priorities for the order of operations. If the last example above had no parentheses, we get a different result: $-3t^2$ does not simplify because the exponent applies to the variable only (do exponents before multiplications).

Simplify expressions of the form $(a^m)^n$

This form shows a **single base with successive powers**. $(a^m)^n$ represents the product of n copies of (a^m) . That is, we need m copies of the base n times over. The result is $(a^m)^n = a^{m \cdot n}$. For an example, consider $(x^5)^2$. This represents the product of 2 copies of (x^5) . Thus $(x^5)^2 = (x^5)(x^5) = x^{10}$ (note how the first law of exponents applies, because now we are multiplying two expressions with the same base x). The shortcut is to simply multiply the powers together to get the simplified form of the answer, $(x^5)^2 = x^{5 \cdot 2} = x^{10}$.

Examples $(y^7)^3 = y^{7 \cdot 3} = y^{21}$

$$(5^2)^6 = 5^{2 \cdot 6} = 5^{12}$$

$$(z^9)^5 = z^{45}$$

Simplify expressions involving two or more of these forms

Some expressions will involve more than one of the exponent laws. You will need to carefully determine which laws to apply. In general, start by simplifying within any parentheses groups. Work your way outward, applying exponents to the factors of each group, and then finish by multiplying the simplified groups together.

Example Simplify $(3x^3y^4)^2$.

Solution This is a combination of the second and third forms: $(ab)^n = a^n b^n$ and $(a^m)^n = a^{m \cdot n}$. The power 2 applies to each factor inside. Since some of these factors already have powers, the exponents are multiplied. Thus $(3x^3y^4)^2 = 3^2(x^3)^2(y^4)^2 = 9x^6y^8$.

Example Simplify $(a^2b)^3(2a^4)$.

Solution Simplify the first group by applying the exponent 3 to each factor.

$$(a^2b)^3(2a^4) = (a^6b^3)(2a^4)$$

Then multiply the result by $(2a^4)$ to get the final answer, $2a^{10}b^3$.

Example Simplify $(5p^4q^3)^2(3pq)^2$.

Solution Each group must be simplified first, using the exponent on the right parenthesis.

$$(5p^4q^3)^2(3pq)^2 = (25p^8q^6)(9p^2q^2) = 225p^{10}q^8$$

The following table summarizes the exponent rules introduced in this section. Use each rule only if the form of the expression exactly matches. The exponents n and m may be any numbers. The variables x and y are used generically and could be any letter or number in the base.

Rules of Exponents		
$x^n \cdot x^m = x^{n+m}$	$(x \cdot y)^n = x^n y^n$	$(x^m)^n = x^{m \cdot n}$

Your Turn!!

Simplify the following exponent expressions.

1. $2^2 \cdot 2^5$

2. $(3^3)^2$

3. $10^7 \cdot 10^{11}$

4. $x^3(x^7)$

5. $(x^3)^7$

6. $x^2 \cdot x^6$

7. $y \cdot y^5 \cdot y^3$

8. $(y^5)^3$

9. $(b^9)^2$

10. $(2x^3)^4$

11. $(3c^4)^2$

12. $(ab)^3$

13. $(10^4 \cdot 10^2)^2$

14. $(x^3)^4$

15. $(-5x^2y)^2$

16. $(10^4)^3 \cdot 10^5$

17. $(-3x^2y)^3$

18. $(p^5)(p^8)$

19. $x^5(-x^3)^2$

20. $(4x^3y^5)^2(3xy^2z)$

21. $(8h^2k^5)(-5hk)$

22. $(-3a^2b)(10a^4b^3)$

23. $(3ab^2)(3a^2b)(3ab)^2$

24. $-(-5a^3b^5)^3$

Section 2.5 Quotient Rule and Negative Exponents

The exponent rules in the previous section cover simplification of variable expressions involving multiplication only. The division operation is closely related to multiplication, and we can also simplify quotients by applying and expanding on the exponent rules.

Simplify exponential expressions using the property $\frac{a^m}{a^n} = a^{m-n}$

According to the multiplicative property for exponents, $a^m \cdot a^n = a^{m+n}$. To multiply numbers with the same base, we add the exponents. Since division is the opposite of multiplication, it makes sense to guess that the rule uses subtraction of the exponents instead. In reality, we are reducing out common factors in the division problem.

Example Simplify $\frac{x^5}{x^3}$.

Solution Write each exponent form out the long way as a repeated multiplication.

$$\frac{x^5}{x^3} = \frac{xxxxx}{xxx} = \frac{\cancel{x} \cancel{x} \cancel{x} xx}{\cancel{x} \cancel{x} \cancel{x}} = \frac{xx}{1} = x^2.$$

The common factors of x reduce out since each x/x is 1. Using the rule, we get the same answer, but more efficiently: $\frac{x^5}{x^3} = x^{5-3} = x^2$.

So to divide expressions with the same variable base, we subtract the exponents. The quotient property can be applied to both numeric and variable expressions, as long as the numerator and denominator have powers of the same base.

Examples

$$\frac{2^7}{2^4} = 2^{7-4} = 2^3 = 8$$

$$\frac{w^{10}}{w} = w^{10-1} = w^9$$

Simplify expressions with exponent zero

The quotient rule for exponents introduces a couple special cases that we will need to understand in order to fully utilize the properties. The first comes up when the numerator and denominator have the same exponent. After subtracting, we will have a **zero exponent**.

Example Simplify $\frac{x^3}{x^3}$.

Solution Apply the quotient rule and subtract the exponents:

$$\frac{x^3}{x^3} = x^{3-3} = x^0$$

But what does x^0 really mean? The answer lies in the reduced form of the fraction.

Alternate Solution Reduce the common factors:

$$\frac{x^3}{x^3} = \frac{xxx}{xxx} = \frac{\cancel{x}\cancel{x}\cancel{x}}{\cancel{x}\cancel{x}\cancel{x}} = \frac{1}{1} = 1.$$

Comparing the two solutions, we must have $x^0 = 1$ for any value of x except 0 (we will not worry about that case until a Calculus class). In general, a 0 exponent means we multiply **no copies** of the base. Since all multiplication starts with 1, and we multiply nothing else, the answer must be 1. If you are still not convinced, look at the pattern for decreasing exponents:

$$x^3 = 1xxx \quad (1 \text{ times } 3 \text{ copies of } x)$$

$$x^2 = 1xx \quad (1 \text{ times } 2 \text{ copies of } x)$$

$$x^1 = 1x \quad (1 \text{ times } 1 \text{ copies of } x)$$

$$x^0 = 1 \quad (1 \text{ times no copies of } x)$$

Although we usually do not write the coefficient 1, it is still there and must be written when there is nothing else.

Examples $w^0 = 1$

$$26^0 = 1$$

$$(5b)^0 = 1$$

Note that the 0 power applies to the whole group.

$$5b^0 = 5 \cdot 1 = 5$$

Here we do have a coefficient (namely 5), but no copies of the variable.

$$(3x)^0 - 10x^0 = 1 - 10(1) = 1 - 10 = -9$$

Simplify expressions with negative exponent

The second case involves **negative exponents**. If the denominator has a higher power than the numerator, we get a negative exponent after subtracting.

Example Simplify $\frac{x^3}{x^8}$.

Solution Apply the quotient rule and subtract the exponents:

$$\frac{x^3}{x^8} = x^{3-8} = x^{-5}.$$

Alternate Solution Reduce the common factors:

$$\frac{x^3}{x^8} = \frac{xxx}{xxxxxxxx} = \frac{\cancel{x} \cancel{x} \cancel{x}}{\cancel{x} \cancel{x} \cancel{x} xxxxx} = \frac{1}{xxxxx} = \frac{1}{x^5}$$

The negative exponent really indicates that we have a fraction with more division than multiplication.

Comparing the two solutions, we see that $x^{-5} = \frac{1}{x^5}$. The negative power refers to a **division** by repeated copies of the variable. In general, $x^{-n} = \frac{1}{x^n}$; replace the negative power by using 1 over the variable with a positive power.

Example Simplify $\frac{c^3}{c^5}$.

Solution Apply the quotient rule and subtract the exponents:

$$\frac{c^3}{c^5} = c^{3-5} = c^{-2}$$

The negative exponent can also be replaced by making a fraction expression:

$$\frac{c^3}{c^5} = c^{-2} = \frac{1}{c^2}$$

Generally, the fraction form with positive exponent is preferred, but there are times that the negative exponent is more convenient.

Examples

$$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$$

$$8^{-1} = \frac{1}{8^1} = \frac{1}{8}$$

$$w^{-4} = \frac{1}{w^4}$$

$$(4y)^{-2} = 4^{-2} y^{-2} = \frac{1}{4^2} \cdot \frac{1}{y^2} = \frac{1}{16y^2}$$

$$x^3 \cdot x^{-7} = x^{3-7} = x^{-4} = \frac{1}{x^4}$$

$$\frac{p^{-5}}{p^{-2}} = p^{-5-(-2)} = p^{-3} = \frac{1}{p^3}$$

In summary, positive exponents refer to a multiplication by copies of the base. The number of copies is given by the exponent.

With a zero exponent, there is nothing to multiply except the coefficient (which may be 1). In fact, any base with a zero power will be equal to 1. Be careful if there are other coefficients. The power 0 will wipe out any part of the base inside parentheses, but coefficients outside the parentheses will remain.

Negative exponents refer to a division by copies of the base. You may think of the negative sign in the exponent becoming the fraction bar if you like. The negative does not really disappear; it only has a different role to serve.

Rules of Exponents		
$x^n \cdot x^m = x^{n+m}$	$\frac{x^n}{x^m} = x^{n-m}$	$(x \cdot y)^n = x^n y^n$
$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$	$(x^m)^n = x^{m \cdot n}$	$x^{-n} = \frac{1}{x^n}$

Scientific notation

Exponents can be used to write very large and very small numbers in a more concise way. Scientific notation is commonly used in many science applications. Astronomy, for example, deals with very large scales, while biology and chemistry often study objects of very small sizes. This section introduces the form of the scientific notation and provides some practice in converting numbers between standard forms and scientific notation. Scientific notation is considered simpler because there is no need to write a long string of 0 digits at the end of a number.

Change numbers written in scientific notation to standard (decimal) form

A typical number written in scientific notation looks like 3.4×10^5 . The number has a decimal part (3.4, in this case) that is multiplied by a power of 10. We can convert back to standard form simply by multiplying out the product. The exponent is done first, of course:

$$10^5 = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 100,000.$$

A positive power of 10 always gives a 1 followed by a number of zeroes equal to the exponent. Now multiply this by the decimal part $3.4 \times 100,000$ is equal to 340,000. Note that it is standard to use “ \times ” for “times” in scientific notation. This is perhaps the only place we will routinely use the “ \times ” symbol in algebra.

Multiplying any number by 10 simply shifts the decimal point one place to the right. For powers of 10, the decimal point is shifted to the right as many places as the value of the exponent.

Example 1.52×10^3 is equivalent to 1,520.

The decimal point was shifted exactly 3 places to the right. Insert zeros to fill all empty places.

Small numbers (close to zero) are written with negative exponents on the base 10. Since a negative power means we need to divide by the factors of 10, the decimal place is shifted to the left instead. $9 \times 10^{-4} = \frac{9}{10^4} = \frac{9}{10,000} = 0.0009$. The decimal place was shifted 4 places to the left.

Example 2.3×10^{-6} becomes 0.0000023.

Note that the decimal point was shifted exactly 6 places to the left.

Write number in scientific notation

To write a number in scientific notation, move the decimal place far enough that there is exactly one (nonzero) digit left of the decimal point. Then multiply by a power of 10, where the exponent is the number of places you moved the decimal point (use a positive exponent if the decimal moved left, and a negative exponent if the decimal moved right). This works to represent the same number, because multiplying by a power of 10 simply moves the decimal point back to where it belongs.

Examples 45,000,000 is the same as 4.5×10^7

$$7,810,000,000 = 7.81 \times 10^9$$

$$540 = 5.4 \times 10^2$$

Examples $0.0000003 = 3 \times 10^{-7}$

(Negative exponents are necessary to write numbers less than 1)

$$0.000764 = 7.64 \times 10^{-4}$$

Multiply and divide numbers using scientific notation

Using scientific notation can simplify calculations involving large numbers. First, convert numbers to scientific notation, and then use exponent rules to reduce the powers of 10.

Example Simplify $(320,000)(50,000,000)$.

Solution $(320,000)(50,000,000)$

$$= (3.2 \times 10^5)(5 \times 10^7) \quad \text{Convert to scientific notation}$$

$$= (3.2)(5) \times 10^5 \times 10^7 \quad \text{Multiply coefficients}$$

$$= 16 \times 10^{12} \quad \text{Add exponents on the base 10}$$

$$= 16,000,000,000,000 \quad \text{Move the decimal point back 12 places}$$

Your Turn!!

Simplify the following exponent expressions.

1. w^0x^2

2. $8w^0x^2$

3. $-9x^0$

4. $10^{-5} \cdot 10^9$

5. $(15d)^0$

6. $5x^0 + (9x)^0$

7. x^{-5}

8. $7t^{-4}$

9. $(4b)^{-3}$

10. $10^4 \cdot (10^{-3})^3$

11. $\left(\frac{2 \times 10^{-4}}{10^{-6}}\right)^2$

12. $\frac{12xy^2}{4xy^{-1}}$

13. $(2a^{-5})^{-2}$

14. $(5x)^{-4}$

15. $\frac{12xy^2}{(4xy)^{-1}}$

16. $(3x^2y)^{-3}$

17. $\left(\frac{xy^2}{z}\right)^{-1}$

18. $\frac{24x^4y^{-3}}{4x^{-4}y^3}$

Write the following numbers in standard (decimal) form.

19. 1.7×10^8

20. 9.17×10^{-5}

21. 4.23×10^6

22. 2.05×10^{-7}

Write the following numbers in scientific notation.

23. 928,000

24. 0.000000103

25. 0.000092

26. 137,100,000,000,000

Chapter 2 Practice Exam

Simplify each of the following.

1. $5 + 4(9)$

2. $5(10 - 4) - 3^2$

3. $7 + 3(1 - 5)$

4. $(-2)^2 - 6(-2) + 8$

5. $5(4x - 2)$

6. $4u - 9v + 7u - 10 - 5v + 6$

7. $-2(6w + 9v - 5)$

8. $(3x^2 - 3x + 6) - (x^2 + 2x - 1)$

Evaluate each of the following expressions with the given values.

9. Evaluate the polynomial $x^2 + 2x - 1$ when $x = 5$.

10. Evaluate the formula $f = \frac{F}{d - F}$ to find the value of f when $F = 30$ and $d = 35$.

Solve the following equations.

11. $5x = 40$

12. $3t - 9 = 12$

13. $5y - 2 = 3y - 20$

14. $2(4z - 3) = z - 14$

15. $10 + 4(3c - 1) = 2(8c - 5)$

16. $2(7x - 2) - 5(2x - 3) = 6(x + 4)$

Scientific Notation:

17. Write 1.36×10^{-4} in standard (decimal) notation.

18. Write 28,000,000,000 in scientific notation.

19. Simplify: $\frac{(1.2 \times 10^{11})(9.5 \times 10^{-7})}{(1.3 \times 10^{-5})}$.

Round your decimal values to the nearest tenth (1 place) and write the simplified answer in scientific notation.

Use the Exponent Rules to simplify each of the following expressions.

20. $m^9 \cdot m^4$

21. $(y^3)^6$

22. $(-3a^5b^3)(5ab^7)$

23. $\frac{4x^{-2}y^3}{14x^5y^{-1}}$

Chapter 2 Answer Section

Section 2.1 Solution Set:

1. 16; 2. -16; 3. 4; 4. 4; 5. 1; 6. 14; 7. 7; 8. 12; 9. 14; 10. -23; 11. 11; 12. 29;
 13. -0.9375; 14. 249; 15. -2.95; 16. -28; 17. 44; 18. 54,872; 19. 1; 20. 100; 21. 14; 22. 29;
 23. -4; 24. 62; 25. 1; 26. 77; 27. 25; 28. 26

Section 2.2 Solution Set:

1. $9b$; 2. $13h$; 3. $3x + 9$; 4. $4x + 1$; 5. $2y - 2x + 12$; 6. $7w + 2t + 11$; 7. $4x^2 + 2x + 8$;
 8. $3y^2 - 6y + 4$; 9. $14x + 10$; 10. $6x - 18$; 11. $12g - 8$; 12. $-8x - 6$; 13. $40a - 30b + 10c$;
 14. $5 - 5x - 15x^2$; 15. $-2x + 21$; 16. $-8 + 16b$; 17. $12p - 6$; 18. $2y + 16$; 19. $13r + 9$;
 20. $19x - 62$; 21. $6f + 13$; 22. $6x + 25$; 23. $4y^2 - 8y + 9$; 24. $-2x^2 + 22x + 29$

Section 2.3 Solution Set:

1. $x = 4$; 2. $x = 6$; 3. $w = -26$; 4. $y = -2$; 5. $r = 22$; 6. $x = -17$; 7. $y = 7$; 8. $x = 3$; 9. $f = 6$;
 10. $c = 7$; 11. $p = -7$; 12. $t = \frac{7}{2}$; 13. $k = 2$; 14. $w = 5$; 15. $x = 7$; 16. $z = 5$; 17. $x = 4$;
 18. $x = -2$; 19. $k = 5$; 20. $x = 3$; 21. $w = 6$; 22. $x = 5$; 23. $y = 6$; 24. $x = 5$; 25. $x = -8$;
 26. $r = \frac{14}{5}$; 27. $x = -\frac{5}{2}$; 28. $p = 7$; 29. $x = 4$; 30. $q = -5$; 31. $v = -\frac{3}{5}$; 32. $x = \frac{2}{3}$; 33. $y = \frac{5}{17}$;
 34. $b = \frac{3}{5}$

Section 2.4 Solution Set:

1. 2^7 or 128; 2. 3^6 or 729; 3. 10^{18} ; 4. x^{10} ; 5. x^{21} ; 6. x^8 ; 7. y^9 ; 8. y^{15} ; 9. b^{18} ; 10. $16x^{12}$; 11. $9c^8$;
 12. a^3b^3 ; 13. 10^{12} ; 14. x^{12} ; 15. $25x^4y^2$; 16. 10^{17} ; 17. $-27x^6y^3$; 18. p^{13} ; 19. $-x^{11}$; 20. $48x^7y^{12}z$;
 21. $-40h^3k^6$; 22. $-30a^6b^4$; 23. $81a^5b^5$; 24. $125a^9b^{15}$

Section 2.5 Solution Set:

1. x^2 ; 2. $8x^2$; 3. -9; 4. 10^4 ; 5. 1; 6. 6; 7. $\frac{1}{x^5}$; 8. $\frac{7}{t^4}$; 9. $\frac{1}{64b^3}$; 10. 10^{-5} ; 11. 4×10^4 ;
 12. $3y^3$; 13. $\frac{a^{10}}{4}$; 14. $\frac{1}{625x^4}$; 15. $48x^2y^3$; 16. $\frac{1}{27x^6y^3}$; 17. $\frac{z}{xy^2}$; 18. $\frac{6x^8}{y^6}$; 19. 170,000,000;
 20. 0.0000917; 21. 4,230,000; 22. 0.000000205; 23. 9.28×10^5 ; 24. 1.03×10^{-7} ;
 25. 9.2×10^{-5} ; 26. 1.371×10^{14}

Chapter 2 Practice Exam Solution Set:

1. 41; 2. 21; 3. -5; 4. 24; 5. $20x - 10$; 6. $11u - 14v - 4$; 7. $-12w - 18v + 10$; 8. $2x^2 - 5x + 7$;
 9. 34; 10. $f = 6$; 11. $x = 8$; 12. $t = 7$; 13. $y = -9$; 14. $z = -\frac{8}{7}$; 15. $c = 4$; 16. $x = -\frac{13}{2}$;

17. 0.000136; 18. 2.8×10^{10} ; 19. 8.8×10^9 ; 20. m^{13} ; 21. y^{18} ; 22. $-15a^6b^{10}$; 23. $\frac{2y^4}{7x^7}$

Chapter 3 - Word Problems

Now, it's time for the word problems. It's not necessary to get worked up and nervous about these. I'm going to show you how to break them down into pictures and equations. Word problems are simply equations written out in a verbal format. It's our job to take the verbal format and turn it into a linear equation.

We use word problems everyday-almost. When you figure your paycheck, it's an equation. x amount of hours times your hourly wage gives you your total paycheck. When you drive 300 miles at 65 miles per hour, you use an equation to find out how long it will take you to get there.

When working word problems, you need to keep some things in mind. The first things are your key words and phrases and common formulas.

To work word problems, we have to create a linear model from the verbal model given. If we can set up an equation that determines what we are looking for, then we substitute our values into the appropriate places. So, where do we start? Well, we can make a step-by-step process that can be used for all word problems.

Step One: Read the Problem!!

Step Two: Draw pictures or diagrams that are helpful.

Step Three: Determine any formulas you will need. These formulas should connect the known quantities with the unknowns.
(Geometry-area, perimeter; Money-interest, sales prices, proportions, percents, etc.)

Sometimes we will use **key words and phrases** to determine what we're looking for. Here are a few examples:

<u>Key Word/Phrase</u>	<u>Equation Form</u>
is	=
of (with a fraction or %)	\times (multiply)
sum of	+
difference	-
x more than y	$x + y$
x less than y	$x - y$
increase	+
decrease	-
twice x or 2 times x	$2x$
n times x	nx

- Step Four: Determine the number of unknowns. Let one of the unknown quantities be represented by a variable, say x , and try to represent all other unknown quantities in terms of x . Be sure you clearly understand what you are letting x represent.
- Step Five: Substitute these unknowns into the formula you found in Step Three. Solve the equation for x .
- Step Six: Check your solution. Does it make sense? For instance, does it make sense to represent distance or time as a negative value?

Section 3.1 Percent Problems

Percent problems usually involve discount problems, sales tax problems, or grading problems. Percent means ‘per 100’. Percents are also interpreted as a part of the whole. If we have 35%, then that means we have 35 parts ‘per 100’ or 35% of the whole. They are another way to write fractions and decimals.

Converting from percent to decimal:

28% means 28 per 100

As a fraction, it is written $\frac{28}{100}$.

As a decimal, we move the decimal point two places to the left:

28% = 28. and, as a decimal, becomes 0.28

12.5% is 0.125 as a decimal

0.62% is 0.0062 as a decimal

Notice that it is easier to convert from a percent to a decimal and then to a fraction because the number of places after the decimal indicates how many zeros in the denominator when converting from decimal to fraction.

$$0.0062 = \frac{62}{10000} \quad 0.125 = \frac{125}{1000} \quad 0.28 = \frac{28}{100}$$

If you are given the decimal and must convert to a percentage, move the decimal two places to the right:

$$0.75 = 75\% \quad 8.27 = 827\% \quad 0.057 = 5.7\%$$

When solving word problems involving percents, it is necessary to convert the percentages to decimals before doing any calculations.

Example A store is giving 30% off the ticketed price. If a shirt has a ticketed price of \$42, how much is the new price after the discount?

Solution To solve this, we would use the formula:
 New Price = Old price – (% discount * Old Price)
 New Price = Old Price (1 - % discount)
 = 42 * 0.70
 = \$29.40

Note that when you are buying this item on sale, you are paying 70% of the original price. The store is taking off 30% of the original price.

We could also write this as $\frac{70}{100} = \frac{x}{42}$ where x represents the part you are paying of the whole.

Using this type of equation is called **proportions**. A proportion is two ratios set equal to each other. It says that 70 is to 100 as what part is to 42? To solve proportions, we use cross multiplication.

$$\begin{aligned}\frac{9}{x} &= \frac{5}{6} \\ 9 \cdot 6 &= 5x \\ 54 &= 5x \\ \frac{54}{5} &= \frac{5x}{5} \\ x &= 10.8\end{aligned}$$

For percent problems, we can use proportions

$$\frac{\%}{100} = \frac{\text{part}}{\text{whole}}$$

For each problem, we must determine which is missing: percent, part, or whole. Use the proportion to fill in what we know and use cross multiplication to solve for the unknown.

Example 27 is 54% of what value?

Solution Percent: 54
 Part: 27
 Notice that 27 is only 54% of some larger number.
 Whole: Unknown, x

$$\frac{\%}{100} = \frac{\text{part}}{\text{whole}}$$

$$\frac{54}{100} = \frac{27}{x}$$

$$54x = 27 * 100$$

$$54x = 2700$$

$$\frac{54x}{54} = \frac{2700}{54}$$

$$x = 50$$

Alternative Solution

Use the key words/phrases above to translate this percentage sentence into an equation:

is becomes '='

of becomes x (multiplication)

'what value' is your unknown-replace with a variable

$$27 = 0.54 * x$$

Divide both sides by 0.54 to get x by itself:

$$x = 50$$

Obviously, this process is much simpler. However, in some word problems, you might still want the proportion.

Example

What percent of \$160 is \$200?

Solution

'what percent' is your unknown, x

of means times/multiply

is means =

$$x \cdot 160 = 200$$

$$160x = 200$$

$$\frac{160x}{160} = \frac{200}{160}$$

$$x = 1.25 = 125\%$$

Example

125% of what value is \$423?

Solution

of means times/multiply

'what value' is your unknown, x

is means =

$$1.25 \cdot x = 423$$

$$1.25x = 423$$

$$\frac{1.25x}{1.25} = \frac{423}{1.25}$$

$$x = \$338.40$$

A common percent application is sales tax. To determine the amount of sales tax, we multiply the sales tax times the price of the item. Check out this quick way to determine the price at checkout.

$$\begin{aligned} \text{Price of Item} &= \$35 & \text{Sales Tax} &= 6.5\% \\ \text{Checkout Price} &= \text{Price of Item} + (\text{Sales Tax} * \text{Price of Item}) \\ &= \text{Price of Item}(1 + \text{Sales Tax}) \\ \text{Checkout Price} &= 35(1 + 0.065) \\ &= \$37.28 \end{aligned}$$

Example The checkout price of a new TV was \$258.48. If the sales tax is 5.5%, what was the sales price?

Solution

$$\begin{aligned} \text{Checkout Price} &= \text{Price of Item}(1 + \text{Sales Tax}) \\ 258.48 &= P(1 + 0.055) \\ 258.48 &= P(1.055) & \text{Divide both sides by 1.055 to isolate P.} \\ P &= \$245.00 \end{aligned}$$

Example A DVD player regularly price at \$199.99 is discounted 20%. What is the new selling price?

Solution

$$\begin{aligned} \text{Selling Price} &= \text{Original Price} - (\text{Original Price} * \text{Discount Amt}) \\ \text{Selling Price} &= \text{Original Price} (1 - \text{Discount Amt}) \\ \text{Selling Price} &= 199.99(0.80) \\ &= \$160.00 \end{aligned}$$

Example A sales person earns \$250 per week plus a 2.5% commission. If the person desires a week's gross pay of \$900, how much merchandise must be sold?

Solution

$$\text{Gross Pay} = 250 + \text{Commission Rate} * \text{Amt of Merchandise Sold}$$

$$900 = 250 + (0.025 * A)$$

$$900 = 250 + 0.025A$$

$$900 - 250 = 250 - 250 + 0.025A$$

$$650 = 0.025A$$

$$\frac{650}{0.025} = \frac{0.025A}{0.025}$$

$$A = \$26,000$$

The person must sell \$26,000 in merchandise to make \$900 for the week.

Percent problems also include problems dealing with interest earned, monthly payments on a loan, or number of years to pay off a loan. It is not required that students memorize these formulas, but you should understand how to use them appropriately.

When getting a loan from a lender, we must pay back both the amount borrowed (the **principal**) plus **interest** for the temporary use of the lender's money. In a simple interest loan everything is paid in one lump sum. For example, if you borrow \$1500 at an interest of 5.0%, then you must pay back 100% of the principal plus 5.0% of the principal = 105% of the principal or $(1.05)(\$1500) = \1575 .

Most conventional loans are paid back in a sequence of monthly payments. In each payment a portion is used to pay the interest owed on the remaining debt and what is left over is used to reduce the debt, i.e., is paid against principal. A conventional loan is characterized by the following four parameters.

1. The initial principal symbolized by P.
2. The annual percentage rate (APR) of interest symbolized by R.
3. The number of years over which the loan is paid off (the period of amortization) symbolized by N.
4. The monthly payment symbolized by M.

To calculate M knowing P, R, and N use the formula:

$$M = \frac{P \cdot R}{12 \cdot \left[1 - \frac{1}{\left(1 + \frac{R}{12}\right)^{12 \cdot N}} \right]}$$

To calculate N knowing P, R and M use the formula:

$$N = \frac{\log \left[1 - \frac{R \cdot P}{12 \cdot M} \right]}{-12 \cdot \log \left[1 + \frac{R}{12} \right]}$$

To calculate P knowing M, R, and N use the formula:

$$P = \frac{12 \cdot M}{R} \left[1 - \frac{1}{\left(1 + \frac{R}{12} \right)^{12N}} \right]$$

In the formula for N, log is the logarithm function, which is found on any scientific calculator. These formulas are rather complicated although they can be easily computed using a spreadsheet program such as Excel or Quatro Pro. With patience they can also be done on a scientific calculator. To illustrate these calculations consider the following three examples.

Example If you borrow \$12,000 at an annual percentage rate of 2.3% to be paid off over 4 years, what is your monthly payment?

Solution $P = \$12,000$, $R = 0.023$ and $N = 4$, so we need to use the formula for M.

It is easier to fill in the values and then take the problem step by step to solve. If you have a graphing calculator, these formulas can be entered all at once as long as you watch the parentheses.

$$M = \frac{P \cdot R}{12 \cdot [1 - 1 \div (1 + R / 12)^{12 \cdot N}]}$$

$$M = \frac{12000 \cdot 0.023}{12 \cdot [1 - 1 \div (1 + 0.023 / 12)^{12(4)}]}$$

$$M = \$261.92$$

We can also percent problems to solve for Percent Tolerance Percent Error. When a measurement is specified on a precision part, such as a diameter of 0.150 in plus or minus 0.001 in, the 0.001 in is called the **tolerance**. This means that any measured diameter between 0.149 in and 0.151 in is acceptable. Another way of specifying the tolerance is the percent tolerance defined by the following formula:

$$\text{Percent Tolerance} = \frac{\text{tolerance}}{\text{specified value}} \times 100\%$$

Again multiplying by 100% changes the fractional tolerance into a percent tolerance. The following examples illustrate percent tolerance.

Example A resistor is rated at $85\Omega \pm 2\Omega$. What is the percent tolerance?

Solution Percent Tolerance = $2\Omega \div 85\Omega \times 100\% = 2.35\%$

Example An acceptable voltage reading is 15.0 V with a percent tolerance of 3%. Express this tolerance in volts.

Solution Tolerance = $3\% \times 15.0 \text{ V} = 0.03 \times 15.0\text{V} = 0.45\text{V}$
So the voltage is $15.0 \text{ V} \pm 0.45 \text{ V}$.

Percent error is similar to percent tolerance. When an actual measurement is made and compared to the “true” or “specified” value, the percent error is defined by the following formula:

$$\text{Percent Error} = \frac{\text{measured value} - \text{specified value}}{\text{specified value}} \times 100\%$$

Usually if the calculated percent error is negative the minus sign is ignored. This is because we are concerned with “how close” we are to the specified value and not whether we are above or below it. The following illustrates a percent error calculation.

Example A part is specified as having a diameter of 0.250 in. The manufactured part measures 0.254 in. What is the percent error?

Solution Percent Error = $(0.254 \text{ in} - 0.250 \text{ in}) \div 0.250 \text{ in} \times 100\% = 1.60\%$

Your Turn!!

Write each of the following numbers as a percent:

1. 0.37 1) _____

2. 0.012 2) _____

3. 2.59 3) _____

4. $\frac{3}{8}$ 4) _____

5. $1\frac{2}{3}$

5) _____

Write each percent as a decimal number:

6. 22.5%

6) _____

7. 211%

7) _____

Write each percent as a fraction reduced to lowest terms:

8. 125%

8) _____

9. $1\frac{5}{6}\%$

9) _____

Solve for the following unknowns:

10. What is 16.5% of 128?

10) _____

11. 125% of \$950 is what amount?

11) _____

12. 53 is what percent of 120?

12) _____

13. 12 is to 80 like 45 is to what number?

13) _____

14. 12 is to 60 like what amount is to 90?

14) _____

15. \$125 is 36% of what amount? (Round to the nearest penny.)

15) _____

16. What percent of $\frac{3}{8}$ is $\frac{5}{16}$?

16) _____

17. A license cost \$120. If the cost increases 7.5%, what is the new cost of this license?

17) _____

18. If sales tax is 5.5%, what would be the check out price of a band saw with a list price of \$289?

18) _____

19. From a 15.0 lb cylinder 1.6 lb of material is removed during machining. What percent of material was removed?

19) _____

20. An electrical resistor is rated at 85 ohms plus or minus 5%. Express this tolerance in ohms.

20) _____

21. A piston is to have a diameter of $0.787 \text{ in} \pm 0.003 \text{ in}$. What is the percent tolerance?

21) _____

22. Specifications call for a pin to be 1.500 in long. If the finished pin measures 1.504 in., what is the percent error?

22) _____

23. If an electric drill usually selling for \$89.95 is on sale at a discount of 25%, what is the new list price?

23) _____

24. A new car is advertised as selling for \$14,220. This price reflects a 9% discount. What was the original (list) price? (Round to the nearest penny.)

24) _____

25. An assembly line is shut down for inspection if the fraction of defective products exceeds 0.5%. If the normal day's production is 10,500 units, at most how many defective units can there be if the line is not to be shut down?

25) _____

26. A salesperson is paid \$380 per week plus a 2.2% commission. What is the person's sale total if the gross pay for a given week is \$1,395? (Round to the nearest penny.)

26) _____

27. What is the monthly payment required to pay off a \$12,000 loan in two years at an annual percentage rate of 2.7%? (Round to the nearest penny.)

27) _____

28. What is the largest amount which can be borrowed over three years at 4.5% APR if the largest affordable monthly payment is \$279? (Round to the nearest ten dollars.)

28) _____

29. How long would it take to pay off \$15,000 at 5.2% APR if the monthly payment is \$450? (Round to the nearest tenth of a year.)

29) _____

30. You can afford 15% of your monthly income of \$2300 on car payments. If the quoted annual percentage rate of the loan is 2.5% over three years, what is the most you can borrow? (Round to the nearest ten dollars.)

30) _____

Section 3.2 Direct and Inverse Variation Problems

If a ratio equals a constant, the two variables **vary directly** or are **directly proportional**. When we say, ‘ y varies directly with x ,’ or ‘ y is directly proportional to x ,’ then the linear model is

$$y = kx$$

where k is the **constant of proportionality** or **rate of change**.

If you think about the distance formula ($d = rt$), notice that r , rate, is a constant of proportionality. The dependent variable is d , distance, because it depends upon the time, t , which you drive. However, your rate of speed, r , stays constant. The formula is seen as, ‘distance varies directly with time.’ Another example is your hourly wages. Let’s use the equation $s = wh$ where s is the salary, w is the hourly wage, and h is the number of hours worked. Again, s is the dependent variable because your salary depends upon the number of hours, h , that you work. And unless you somehow pull a raise out of your boss, your hourly wage, w , remains constant. So, ‘salary is directly proportional to the number of hours worked.’

Example P varies jointly with r and s . If $P = 16$ when $r = 5$ and $s = -8$, find P when $r = 2$ and $s = 10$.

Solution Let k represent the constant of proportionality. The representation of the relationship above is $P = krs$. The information given does not provide the value for k , but we can find it.

$$\begin{aligned}
 P &= krs \\
 16 &= k(5)(-8) \\
 16 &= -40k \\
 k &= 16/(-40) = 2/(-5) = -0.4
 \end{aligned}$$

Now we can complete the equation: $P = -0.4rs$
 And, the final part of the problem:

$$\begin{aligned}
 P &= -0.4rs \\
 P &= -0.4(2)(10) \\
 P &= -8
 \end{aligned}$$

There are other types of variations we can work with. y can also be inversely proportional to x . This is represented by $y = k/x$.

Example R is inversely proportional to the square of I. If I = 25, then R = 100. Find the constant of proportionality.

Solution $R = \frac{k}{I^2}$ Equation

$$100 = \frac{k}{25^2}$$

$$625 * 100 = k$$

$$k = 62,500$$

Often a quantity depends on more than one variable. For example, the pressure of a gas depends on the amount of gas present (the number of “moles”, n), the absolute temperature (T), and the volume occupied by the gas (V). The relationship between these variables is that the pressure of a gas varies as the product of the amount of gas and the absolute temperature and inversely as the volume occupied by the gas. The phrase, “varies as the product”, means the same thing as varies directly as the product. Thus, the relationship is described by the following formula.

$$P = \frac{RnT}{V}$$

Here, by standard usage, the proportionality constant is called R .

Problems involving variation are often of the form of providing two different sets of data with an unknown quantity in one of the data sets. The simplest way to solve this kind of problem is to convert it into a proportion as the following examples illustrate.

Example A drive gear having 80 teeth and rotating at 40 rpm meshes with a second gear having 16 teeth. How fast does the second gear rotate?

Solution This is a proportion problem since for two meshing gears the number teeth per minute that pass the contact point must be the same for both gears. Let the number of teeth be N and the rate of rotation (measured in rpm) be r , then $N \cdot r = k$ (k a constant). Stated differently, $\frac{Nr}{N} = r = \frac{k}{N}$. This means that the rate of rotation for meshing gears is inversely proportional to the number of teeth. The fewer teeth, the faster the gear rotates. Let $N_1 = 80$, $r_1 = 40$ rpm and $N_2 = 16$. The use of subscripts enables us to label the values that belong together in the same data set. The unknown quantity we seek is C , the rotation rate of the second gear. Since $r_1 = \frac{k}{N_1}$ and $r_2 = \frac{k}{N_2}$, by taking the ratio of r_2 to r_1 the constant k drops out.

$$\frac{r_2}{r_1} = r_2 \div r_1 = \frac{k}{N_2} \div \frac{k}{N_1} = \frac{k}{N_2} \cdot \frac{N_1}{k} = \frac{N_1}{N_2}$$

$$\frac{r_2}{r_1} = \frac{N_1}{N_2}$$

Since r is inversely proportional to N , the subscripts on N are opposite (the numerator and the denominator are switched) to those on r . Plugging in the values for the variables gives the proportion, $\frac{r_2}{40 \text{ rpm}} = \frac{80}{16}$, which we solve for r_2 .

$$\frac{r_2}{40 \text{ rpm}} = \frac{80}{16}$$

$$16r_2 = 40 \text{ rpm} \cdot 80$$

$$\frac{16r_2}{16} = \frac{40 \text{ rpm} \cdot 80}{16} = 40 \text{ rpm} \cdot \left(\frac{80}{16}\right)$$

$$r_2 = 40 \text{ rpm} \cdot 5 = 200 \text{ rpm}$$

Your Turn!!

1. A drive gear having 100 teeth and rotating at 40 rpm meshes with a second gear having 20 teeth. How fast does the second gear rotate?
2. If y is directly proportional to x and $x = 10$ when $y = 4$, what is x when $y = 12$?
3. If x is inversely proportional to y and $x = 24$ when $y = 3$, what is x when $y = 6$?

4. If y varies directly as the product of x^2 and t , what is t when $y = 10$ and $x = 5$, if y is 15 when $x = 3$ and $t = 5$?

For the problems 5 - 7, use the information that the pressure of a gas, P , varies directly as the absolute temperature T and inversely as the volume, V .

5. If P is 10.5 psi when T is 400 K and V is 3.5 L, what is P when T is 380 K and V is 3.0 L?

6. If P is 25.0 psi when T is 400 K and V is 6.00 cubic feet, what is T when P is 20.0 psi and V is 9.00 cubic feet?

7. If P is 1.0 atmosphere when T is 300 K and V is 22.0 L, what is V when T is 450 K and P is 3.2 atmospheres?

8. The resistance, R , of a wire varies directly as the length, l , and inversely as the diameter, d , squared. If R is 3.64 ohms when $l = 5.0$ m and $d = 1.25$ mm, what is the resistance of a wire made of the same material that is 20.0 m long and has a diameter of 2.50 mm?

Section 3.3 Other Types of Word Problems

Number Problems

Number problems involve translating verbal forms involving only information about numbers into symbolic forms.

Example Find a number such that 7 times a number is 3 more than the number.

Solution Let x represent the unknown, the number.

How would you translate the words into an equation?

7 times a number becomes $7x$

is becomes '='

3 more than the number becomes $3 + x$

Replace the words with the translations:

$$7x = 3 + x$$

$$\begin{aligned} \text{Solve for } x: \quad 7x &= 3 + x \\ 7x - x &= 3 + x - x \\ 6x &= 3 \\ (6/6)x &= 3/6 \\ x &= 1/2 \end{aligned}$$

Example

Find three consecutive odd numbers such that 4 times the first minus the third is the same as the second. Write an equation and solve.

Solution

Construct 3 consecutive odd numbers:

$$\begin{aligned} 1^{\text{st}} \text{ number: } & x \\ 2^{\text{nd}} \text{ number: } & x + 2 \\ 3^{\text{rd}} \text{ number: } & x + 4 \end{aligned}$$

Note: These consecutive numbers will also work for consecutive even numbers. Consecutive integers (not specified as odd or even) are $x, x + 1$, etc.

Rewrite the verbal model into an equation:

4 times the first becomes $4x$
 minus the third becomes $-(x + 4)$
 is becomes '='
 the second becomes $x + 2$

Replace the words with the translations:

$$4x - (x + 4) = x + 2$$

$$\begin{aligned} \text{Solve for } x: \quad 3x - 4 &= x + 2 \\ 3x - x - 4 &= x - x + 2 \\ 2x - 4 &= 2 \\ 2x - 4 + 4 &= 2 + 4 \\ 2x &= 6 \\ (2/2)x &= 6/2 \\ x &= 3 \end{aligned}$$

The first integer is 3, the second integer is 5, and the final integer is 7.

Geometric Problems

Geometric problems involve area, perimeter, circumference and volume.

Example A rectangle is three times as long as it is wide. The perimeter is 112 cm. What are the rectangle's dimensions?

Solution The key word here is perimeter because it indicates that this is a geometric problem. The word 'perimeter' tells us what type of geometric problem we have. The problem also indicates that the shape is a rectangle.

To find the perimeter of the rectangle, we add up the lengths of all four sides. The formula for the perimeter is $P = 2L + 2W$ where P is perimeter, L is length, and W is the width.

In this problem, the unknown is the dimensions of the rectangle. What do we know about the rectangle? We know that P, the perimeter, is 112 cm. We also know that the length, L, of the rectangle, is three times ($3x$) the width, W.

$$\begin{aligned} \text{Therefore,} \quad 112 &= 2(3W) + 2W && \text{where } L = 3W \text{ (3 times the width)} \\ 112 &= 6W + 2W \\ 112 &= 8W \\ 112/8 &= (8/8)W \\ W &= 14 \text{ cm} \end{aligned}$$

If $L = 3W$, then the length is 3 times 14 which is 42 cm.

Distance Problems

The most common distance formula is **distance = rate * time**.

Example A car leaves a city traveling at 60 mph per hour. How long will it take a second car traveling at 80 mph per hour to catch up to the first car if it leaves 2 hours later?

Solution Let's make a table. The table will compare the distance, rate, and time for both cars. Notice that the distance both cars travel is the same. Our verbal equation will look like:

$$\text{Distance Car 1 Travels} = \text{Distance Car 2 Travels}$$

	Rate	Time	Distance
Car 1	60 mph	x	$60x$
Car 2	80 mph	$x - 2$	$80(x - 2)$

Car 2 has time $x - 2$ because it will take 2 hours LESS for the second car to reach the destination. Imagine you leave at 9 a.m. for Kansas City. You arrive at 1:00. Your friend leaves at 11:00 and still arrives in Kansas City at 1:00. You both arrived at the same time, but it took your friend 2 hours LESS THAN you to get there.

Remember the verbal equation for this problem. Now that I have the distance for both cars, I can substitute my unknowns into the formula.

$$\begin{aligned} 60x &= 80(x - 2) \\ 60x &= 80x - 160 \\ -20x &= -160 \\ x &= 8 \end{aligned}$$

It will take the first car 8 hours and the second car 6 hours to catch up with each other.

Mixture Problems

Mixture problems involve combining two elements in order to make a third. An example would be a store owner who has a bin of peanuts and a bin of cashews. He wants to make a third bin that contains a mixture of the peanuts and cashews together. He would use a mixture problem formula in order to determine how much of each type he needs in order to have the correct combination at the given price per pound. These problems can also be related to chemical solutions, as in the following examples.

Example A chemical stockroom has a 20% alcohol solution and a 50% solution. How many deciliters of each should be used to obtain 90 deciliters of a 30% solution?

Solution Let's make a table to sort out our information. We can determine that the unknowns are how much we took from the 20% solution and the amount taken from the 50% solution.

Before we make the table, let's look at what we have. There is some quantity taken from one mixture plus a quantity taken from a second mixture to obtain a third mixture. Therefore, our verbal equation is:

$$\text{Amount of Mix A} + \text{Amount of Mix B} = \text{Amount of Mix C}$$

	Mixture A	Mixture B	Mixture C
% solution	0.20	0.50	0.30
Amt of deciliters	x	$90 - x$	90
Total Taken	$0.20x$	$0.50(90-x)$	$0.30 * 90 = 27$

How do we know the amounts of deciliters from A and B? We know that the total amount in the final product is 90 deciliters. Therefore, if we took (for example) 30 deciliters from A we know 60 deciliters were taken from B because the total amount taken is 90 deciliters. Therefore, we let x represent the amount taken from A and the difference between what the end product contains and what we have taken from A is what was taken from B.

Now, substitute the table values into the verbal equation.

$$\text{Amount of Mix A} + \text{Amount of Mix B} = \text{Amount of Mix C}$$

$$\begin{aligned} 0.20x + 0.50(90-x) &= 27 \\ 0.20x + 45 - 0.50x &= 27 \\ -0.30x &= -18 \\ x &= 60 \end{aligned}$$

60 deciliters were taken from the 20% solution and 30 deciliters were taken from the 50% solution.

Example How many grams of 10% tin solder must be added to 800 g of 15% tin solder in order to make a final solder which is 12% tin?

Solution Let x stand for the amount of 10% tin solder. When we combine the x grams of the 10% tin solder with the 800 g of 15% solder, the final 12% tin solder must contain $800 + x$ grams. The following table is set up in a different manner than the first. You might find this table serves you as a more useful tool in putting the information together.

	10% Tin		15% Tin		12% Tin
Amount of Solder	x	+	800 g	=	$x+800$ g
% Tin	10%		15%		12%
Amount of Tin	$0.10x$		$0.15(800) = 120$		$0.12(x+800)$
	$0.10x$	+	120	=	$0.12(x+800)$

The tin in the final 12% solder must all have come either from the 10% tin solder or the 15% tin solder. This translates into the equation, $0.10x + 120 = 0.12(800 + x)$, which we solve for x .

$$0.10x + 120 = 0.12(800 + x)$$

$$0.10x + 120 = 0.12(800) + 0.12x$$

$$0.10x + 120 = 96 + 0.12x$$

$$0.10x - 0.10x + 120 = 96 + 0.12x - 0.10x$$

$$120 = 96 + (0.12 - 0.10)x$$

or

$$96 + 0.02x = 120$$

$$-96 + 96 + 0.02x = -96 + 120$$

$$0.02x = 24$$

$$\frac{0.02x}{0.02} = \frac{24}{0.02}$$

$$x = 1200$$

This answer can be verified by plugging it into the previous table, where indeed it is true that $120 + 120 = 0.12(2000)$.

Your Turn!!

1. The sum of two consecutive odd integers is 444. What are the two numbers?
2. A rectangle is three times as long as it is wide. Its perimeter is 48 in. What are the rectangle's dimensions?
3. It takes you four hours to drive to a city. If you had driven 15 mph faster, you would have gotten there in only three hours. How far did you drive?
4. John's age is one year less than twice Mary's age. Their combined age is 74 years. How old are they?
5. In Sarah's pocket there is \$2.51 in change. There is one more dime than quarters, but one less dime than twice the number of nickels. The number of pennies is three less than twice the number of dimes. How many of each type of coin are there?
6. How many grams of 16% tin solder must be added to 200 g of 25% tin solder to make a final solder which is 20% tin?

Chapter 3 Practice Exam

1. Find percent notation for $\frac{7}{8}$.
2. Find fractional notation for 9.75%.
3. Find the decimal notation for 20%.
4. Solve: 175 is what percent of 100?
5. Solve: 80 is what percent of 40?
6. Solve: What is 55% of 262?
7. The sales tax rate in a city is 8.2%. Find
 - a) the tax charged on a purchase of \$238
 - b) the total cost
8. Of all the people that attend movies, 67% are in the 12-29 age group. At one theater, 600 people attended a showing of a certain movie. How many were in the 12-29 age group?
9. Find the missing values:

Marked Price	Rate of Discount	Discount	Sale Price
\$109.00		\$21.80	

10. A '40% off sale' begins today at Wanda's Women's Wear. What is the sale price of women's wool coats normally priced at \$250?

11. The population of a state was 1,563,000 with 44.5% represented by a minority. What is the best estimate of the minority population in this state?
12. Joe took an exam with 16 questions. He got 3 of the questions wrong. What percent of the questions did he get *right*?
13. A rectangle is three times as long as it is wide. Its perimeter is 24 in. What are the rectangle's dimensions? Note: The perimeter is the sum of all four sides of a rectangle.
14. A house cleaning service charges a fee of \$50 a visit and then \$10.75 per hour for cleaning a home. The Robert's had the service clean their house last spring and it cost them \$211.25. How long did the cleaning service take to clean the Robert's house?
15. Determine n : $\frac{12}{n} = \frac{9}{16}$
16. Most mammals breathe about once for every 4 heartbeats. If a runner after finishing a race has a heart rate of 125 beats per minute, how many breaths is the runner taking in one minute?
17. The time it takes to complete a task varies inversely with the number of people performing the task. If Jim's lawn service using two mowers, they can cut a large lawn in 5.5 hours, how long would it take to cut the grass on the same lawn if 7 mowers were used?
18. If y varies jointly as x and the square of z , and if $y = 192$ when $x = 12$ and $z = 9$, then what is y when $x = 2$ and $z = 16$?
19. The resistance, R , of a wire varies directly as the length, l , and inversely as the diameter, d , squared. If R is 1.50 ohms when $l = 2.0$ m and $d = 4.0$ mm, what is the resistance of a wire made of the same material that is 10.0 m long and has a diameter of 5.0 mm?
20. It takes you three hours to drive to a city. If you had driven 10 mph slower, it would have taken you 36 minutes longer. How far did you drive?

Chapter 3 Answer Section

Section 3.1 Solution Set:

1. 37%; 2. 1.2%; 3. 259%; 4. 37.5%; 5. 166.67%; 6. 0.225; 7. 2.11; 8. $1\frac{1}{4}$; 9. $\frac{11}{600}$
10. 21.12; 11. \$1187.50; 12. 44.17%; 13. 300; 14. 18; 15. \$347.22; 16. 83.33%
17. \$129; 18. \$304.90; 19. 10.67%; 20. $85\sum\forall 4.25\sum$; 21. $\forall 0.38\%$; 22. 0.27%
23. \$67.47; 24. \$15,626.37; 25. 52; 26. \$46,136.36; 27. \$514.18; 28. About \$9380
29. 3.0 years; 30. About \$11,950

Section 3.2 Solution Set:

1. 200 rpm; 2. $x = 30$; 3. $x = 12$; 4. $t = 1.2$; 5. 11.6 psi; 6. 480 K; 7. 10.3 L; 8. 3.64 ohms

Section 3.3 Solution Set:

1. 221 and 223; 2. 18 in by 6 in; 3. 180 miles; 4. John is 49, Mary is 25;
5. six quarters, seven dimes, four nickels, and eleven pennies; 6. 250g

Chapter 3 Practice Exam Solution Set:

1. 87.5%; 2. $\frac{39}{400}$; 3. 0.20; 4. 175%; 5. 200%; 6. 144.1; 7. a) \$19.52, b) \$257.52;
8. 402 people; 9. 20%, \$87.20; 10. \$150; 11. 695,535 people; 12. 81.25%;
13. 3 in x 9 in; 14. 15 hours; 15. 21.3; 16. 31.25 breaths; 17. 1.57 hours;
18. 101.14; 19. 4.80 ohms; 20. 180 miles

Chapter 4 - Statistics

Statistics is the science of collecting, organizing, analyzing, presenting, and interpreting data.

The data that is used in statistics can come to us from many sources. We can either do the data collection ourselves or the data can be obtained from some other source like a newspaper or one of the various levels of government or from a business that collects information about its customers.

There are many different ways in which we can talk about data. We will look at two major classifications of data: qualitative versus quantitative and discrete versus continuous.

Data can be considered to be either qualitative or quantitative. Qualitative data is data that is a name or reference to something to someplace or is a characteristic of something. Examples of qualitative data are a person's hair color or eye color, type of car driven, the letter grade received in a class or the objects shape. Qualitative data can also be numeric; that is, it can contain numbers. For example a zip code is a qualitative data value because it is a reference to a geographical location in the United States. A ranking of 1 through 10 is also qualitative in nature because it is a reference to how well something is being done. Quantitative data is data that is numerical and for which it makes sense to add, subtract, multiply or divide data values. Some examples of quantitative data are the speeds at which cars pass a certain point on a road, the height of a male human, and the number of 8-ounce cups of coffee a person buys in a day. Zip codes and phone numbers are numeric in nature but it would not make any sense to do arithmetic on these numbers and that is why they are qualitative.

Quantitative data can be further defined as being either discrete or continuous. Discrete data is data that can be counted. Some examples of discrete data are how many commercials there are in one hour of television programming, how many cars pass a given point on a road in a day, and how many letters are mailed to a certain zip-code in a week. Continuous data is data that can be measured. Some examples of continuous data are the length of a television commercial, how fast the car are travelling past a given point on a road, and the weight of the letters that are mailed to a certain zip code in a week.

Data collection is the hard part of doing a statistical analysis. Before we begin we must have a data collection plan. That is, we must consider what kind of information we are trying to gather and how are we going to collect this information. We can collect data by direct observation or by means of a survey or we can get our data from another source. Direct observation means that we do the actual measurement or count of the subjects. If we are interested in the temperature at which certain liquids boil then we must be there to insert the thermometer into the liquid and read the temperature at which the liquid boils. If we are going to use a survey then we must design the survey in such a way that the questions we ask are not misleading in nature, have a limited number of possible responses, and are not overly burdensome to complete. If we get our

data from some other source we must ensure that the source is trustworthy and that the data is what we wanted and fits within the framework of the analysis we are doing.

Section 4.1 Organizing Data

When we perform a statistical experiment, doing something that generates a piece of data, we are collecting information. If we only perform the experiment once, we have only one piece of data. What we would probably do is to repeat the experiment several times to see if we are getting the same results or if there are differences from trial to trial. As we keep repeating the experiment, we see that the amount of data that we have keeps increasing in size. The problem we have is how do we manage or organize the data to make the information meaningful to ourselves and to others.

One of the easiest ways to organize data is by means of a frequency distribution. A frequency distribution is a table that pairs classes, the attributes or the numbers, and their frequency. Frequency is the number of times that an experiment results in a particular class as an outcome. The classes can be either qualitative or quantitative. If the classes are qualitative then they will be either the names or the labels or the attributes that we were looking for in the experiment. If the classes are quantitative they can be either discrete values or continuous values. Sometimes when we have a lot of numerical data we may want to group some of the classes together and form a grouped frequency distribution. Let's look at each one of these types of distributions.

Section 4.1.1 Qualitative Frequency Distribution

EXAMPLE:

You are taking Introduction to College Mathematics and wonder how other people in your program have done who have taken the course from the same instructor. You ask 50 people what they got and record the following letter grades:

B	B	A	D	D	C	BC	AB	D	B
C	AB	AB	BC	C	B	AB	B	A	C
AB	BC	C	B	BC	AB	BC	BC	BC	BC
D	C	D	B	BC	B	C	B	BC	B
BC	B	B	B	BC	B	BC	C	C	C

What we would like to do is to form a frequency distribution based upon the grade received. To form a qualitative distribution we do the following:

1. List each name or attribute
2. Make a tally mark next to each name as we record that data value
3. Total the tally marks to get the frequency

In the above example, we want to create a table with 3 columns. The first column would contain the list of possible grades in some reasonable order: in this case from highest to lowest. The second column is where we do our tallying. Every letter grade in our data gets one tally mark next to the corresponding grade. The third column is the frequency or the number of tally marks that occur for a particular grade. For this problem our chart when complete should look like this.

Grades	Tally	Frequency
A		2
AB		6
B		14
BC		13
C		10
D		5
F		0

A good way to check that all of the data has been included in the table is to add the frequencies together and see if the total is the same as the number of data values we had to begin with. In this case we have a total of 50 which is how many data values with which we began.

Section 4.1.2 Quantitative Ungrouped Frequency Distribution

Quantitative frequency distributions can be created for either discrete data or continuous data. We also have the options of creating either grouped or ungrouped frequency distributions for quantitative data. An ungrouped quantitative frequency distribution is a frequency distribution in which every data value is its own class or result of the experiment. We would use an ungrouped distribution if the number of outcomes or data values that are possible is relatively small, the number of outcomes is limited, not more than 10 or so. Let's look at an example of an ungrouped frequency distribution.

EXAMPLE:

Traffic Control has been monitoring the number of cars that cross the intersection of Burgoyne and Washington during non-rush hour periods during the day. Over the last several days they have recorded the following 140 numbers of cars crossing that intersection during non-rush hour daylight hour intervals.

20	19	20	23	23	17	16	19	20	22	22	16
17	20	19	21	17	23	24	25	21	16	25	24
22	19	21	23	23	25	19	23	24	23	22	21
23	21	20	25	17	23	15	19	16	25	25	21
23	17	16	17	23	22	17	24	19	23	17	25
19	22	17	24	25	22	16	17	23	16	17	17

24 20 16 20 18 20 19 18 18 16 23 25
 24 16 21 19 16 23 23 20 16 21 21 17
 15 21 17 22 22 23 18 25 20 24 15 24
 24 25 17 20 21 23 21 24 19 22 24 20
 18 23 15 19 17 22 21 22 22 24 19 24
 15 20 15 20 20 25 20 24

To get a handle on this information we will form an ungrouped frequency distribution. To form this distribution we must do the following:

1. Determine the low and the high value
2. List each data value in order from low to high or from high to low
3. Make a tally mark next to each number in our list for each time it appears in the data
4. Total the tally marks to get the frequency

If you noticed, this is a lot like what we did when we formed a qualitative distribution. The difference is that instead of labels we are using numbers. We are going to create a table with 3 columns. The first column would contain the number of cars per hour reported listed in a reasonable order: in this case from lowest to highest. The second column would be where we do our tallying. Every number of cars per hour in our data set gets one tally mark next to the number of cars per hour possible. The third column is the frequency or the number of tally marks that occur for a particular number of cars per hour. For this problem our chart when complete should look like this.

Cars / hour	Tally	Frequency
15		6
16		12
17		16
18		5
19		13
20		16
21		13
22		13
23		19
24		15
25		12

A good way to check that all of the data has been included in the table is to add the frequencies together and see if the total is the same as the number of data values we had to begin with. In this case we have a total of 140 for the sum of the frequencies which is how many data values with which we began.

Section 4.1.3 Quantitative Grouped Frequency Distribution

As we said above quantitative frequency distributions can be created for either discrete data or continuous data and we also have the options of creating either grouped or ungrouped frequency distributions for quantitative data. A grouped quantitative frequency distribution is a frequency distribution in which several data values or a range of data values form a class. We would use a grouped distribution if the number of outcomes or data values that are possible is large, more than 10 or so. Before we look at an example, we have to know about some guidelines to structure the classes that will be used in the distribution.

Here are just a couple on notes about classes before we look at how to construct the distribution itself.

- There should be between approximately 5 and 20 classes.
- If possible the class width, the difference of the upper end of the class and the lower end of the class, should be an odd number because this will guarantee that the class midpoints are whole numbers instead of decimals or fractions.
- All classes should be the same width.
- The classes must be mutually exclusive. This means that no data value can fall into two different classes.
- The classes must be all inclusive or exhaustive. This means that all data values must be included. The classes must be continuous; that is, there are no gaps in a frequency distribution. Classes that have no values in them must be included unless it's the first or last class in which that class is dropped.

We can now create a grouped frequency distribution by following the below steps:

1. Select an appropriate number of classes to ensure that the distribution will contain between approximately 5 and 20 classes.
2. Determine the largest and the smallest data values in the data set.
3. Determine the range. The range is the difference of the largest minus the smallest data value.
4. Determine the class width. The class width is the range divided by number of classes. Class width should be a whole number, if possible. If the division resulted in a fraction we can round up to the next whole number. If the division resulted in a whole number we will have to add an extra class.
5. Pick a starting point that is less than or equal to the minimum value, the lowest number in the data set. This is the lower limit of the first class. To determine the next lower limit, add the class width to the previous lower limit and keep going until you have a number bigger than the largest value in the data set.
6. To determine the upper limits of the classes we first list in a column all of the lower limits. We then look at the last digit of the number. If our data is whole numbers then the last digit is units and we subtract 1 from the next lower limit to determine the previous upper limit. If our data is decimal we subtract the correct place with a one from the next lower limit to determine the previous upper limit and keep going until you have a

number bigger than the maximum value in the data set. Delete any classes above an upper limit that is bigger than the maximum value in the data

7. Determine the class boundaries. A boundary is point half way between the upper limit of a class and the lower limit of the next class. The boundaries ensure that no data value can be in two classes. Sometimes it is not necessary to determine the boundaries.
8. Tally the data by determining into which class each data value belongs.
9. Determine the frequencies by counting the tally marks.

Let us now look at an example of how to construct a grouped frequency distribution. We will use the Traffic Control example from above to compare how a grouped frequency distribution looks compared to an ungrouped distribution.

EXAMPLE:

Traffic Control has been monitoring the number of cars that cross the intersection of Burgoyne and Washington during non-rush hour periods during the day. Over the last several days they have recorded the following 140 numbers of cars crossing that intersection during non-rush hour daylight hour intervals.

20	19	20	23	23	17	16	19	20	22	22	16
17	20	19	21	17	23	24	25	21	16	25	24
22	19	21	23	23	25	19	23	24	23	22	21
23	21	20	25	17	23	15	19	16	25	25	21
23	17	16	17	23	22	17	24	19	23	17	25
19	22	17	24	25	22	16	17	23	16	17	17
24	20	16	20	18	20	19	18	18	16	23	25
24	16	21	19	16	23	23	20	16	21	21	17
15	21	17	22	22	23	18	25	20	24	15	24
24	25	17	20	21	23	21	24	19	22	24	20
18	23	15	19	17	22	21	22	22	24	19	24
15	20	15	20	20	25	20	24				

To form the distribution we will follow the above steps.

Step 1. Select the number of classes.

Let us choose 5 classes.

Step 2. Determine the high and the low data values.

The highest value is 25 and the lowest value is 15

Step 3. Determine the range.

The range is $25 - 15 = 10$

Step 4. Determine the class width.

The class width is $10 \div 5 = 2$. We will have to add a class.

Step 5. Pick a starting point: the first lower limit.

We will start at 15, the lowest value and our first lower limit

The rest of the lower limits are 17, 19, 21, 23, 25, and 27. Wait. That gives us six classes. That means that 5 classes will not work with a width of 2 so we need that extra class.

Step 6. Determine upper limits. We list the lower limits in a column.

15
17
19
21
23
25
27

We note that we have whole numbers so we subtract 1 from the next lower limit to get the previous upper limit. Doing that gives us the following list of classes with lower and upper limits.

15 - 16
17 - 18
19 - 20
21 - 22
23 - 24
25 - 26

We can delete the 27 lower limit because we will have no data values that fall in that class.

Step 7. Determine the class boundaries.

Because the data is discrete we do not need boundaries, but if we did they would be the halfway point between an upper limit and the next lower limit. In this case the boundaries are 14.5, 16.5, 18.5, 20.5, 22.5, 24.5, and 26.5.

Step 8 and Step 9 Tally and determine frequencies

We create the following chart.

Cars / hour	Tally	Frequency
15 – 16		18
17 – 18		21
19 – 20		29
21 – 22		26
23 – 24		34
25 - 26		12

We have created a grouped frequency distribution. Compare this to the ungrouped frequency distribution example above.

We will now form another grouped frequency distribution that is a little more involved because the data we collected has a decimal place in it. The process we will follow is as listed in the above 9 steps.

EXAMPLE:

A student nurse has taken the temperature of 108 people that have visited the clinic in which she works. Their temperatures are listed below. Create a grouped frequency distribution for the temperatures using 7 classes.

100.1	103.3	97.5	100.0	101.5	101.8	100.2	102.1	98.5
99.1	98.7	100.6	99.0	97.5	99.2	99.1	101.5	103.0
100.9	99.3	103.4	99.9	98.8	97.1	104.0	101.1	98.0
100.8	97.8	103.4	101.1	102.6	99.8	102.9	103.1	103.7
100.0	103.0	98.7	103.7	103.5	97.3	98.3	99.4	101.1
98.9	98.4	100.4	99.1	103.2	103.5	103.7	103.4	103.0
98.2	99.7	103.4	103.6	98.7	97.1	98.2	97.8	102.4
102.4	102.9	103.8	100.2	101.8	100.4	99.2	101.6	100.9
97.3	97.1	97.8	102.6	100.7	97.8	98.8	103.2	98.7
102.5	97.8	99.5	97.0	103.2	100.6	99.3	97.87	102.5
97.6	98.4	103.1	97.9	99.4	99.3	103.1	103.4	102.4
98.0	102.1	103.9	98.7	97.1	100.6	100.9	103.4	100.2

To form the distribution we will follow the above 9 steps outlined above.

Step 1. Select the number of classes.

We were asked to use 7 classes.

Step 2. Determine high and low.

The highest value is 104.0 and the lowest value is 97.0

Step 3. Determine the range.

The range is $104.0 - 97.0 = 7$

Step 4. Determine the class width.

The class width is $7 \div 7 = 1$. A whole number means we have to add a class

Step 5. Pick a starting point.

We will start at 97.0, the lowest value and our first lower limit. The rest of the lower limits are 98.0, 99.0, 100.0, 101.0, 102.0, and 103.0.

Step 6. Determine upper limits.

We list the lower limits in a column.

97.0

98.0

99.0

100.0
 101.0
 102.0
 103.0
 104.0
 105.0

We note that we have a tenth of a unit so we subtract 0.1 from the next lower limit to get the previous upper limit. Doing that gives us the classes

97.0 – 97.9
 98.0 – 98.9
 99.0 – 99.9
 100.0 – 100.9
 100.0 – 100.9
 102.0 – 102.9
 103.0 – 103.9
 104.0 – 104.9

We can delete the 105.0 class because we will have no data values that fall in that class.

Step 7. Determine the class boundaries.

Because the data is continuous we may need boundaries. The boundaries are the halfway points between an upper limit and the next lower limit. In this case the boundaries are 96.95, 97.95, 98.95, 100.95, 101.95, 102.95, and 103.9.

Step 8 and Step 9 Tally and determine frequencies

These are given in the completed chart below.

Temperature	Tally	Frequency
97.0 – 97.9		17
98.0 – 98.9		16
99.0 – 99.9		15
100.0 – 100.9		16
101.0 – 101.9		8
102.0 – 102.9		11
103.0 – 103.9		24
104.0 – 104.9		1

Section 4.1.4 Cumulative Frequency Distribution

A cumulative frequency distribution is a distribution that accumulates, adds-up the frequencies, up to and including a specific class. This type of distribution works well when the data can be ordered in some meaningful way. It is particularly useful when the data is numerical or when the data occurs over time. To create a cumulative frequency distribution we must first create a frequency distribution and then we add a new column. The new column is the cumulative

frequencies. The first entry in the cumulative column is the frequency for the first class. The second entry in the cumulative column is the total of the first two frequencies. The third entry in the cumulative column is the total of the first three frequencies. We keep going until we have added together all of the frequencies. This number should check with the number of data values we began with when we started the problem.

EXAMPLE:

In the last example we created a grouped frequency distribution for the temperatures of 108 people who visited a clinic and had their temperature taken by a student nurse. What we would now like to do is to create a cumulative frequency distribution for that data set. We can use what we have created in the last example by adding a new column to the end of the previous table called cumulative frequency. The first row of the table is the same as frequency. The second row is the sum of the first and second frequencies. The third row is the sum of the first three frequencies and so on until we get to the end of the table.

Temperature	Tally	Frequency	Cumulative Frequency
97.0 – 97.9		17	17
98.0 – 98.9		16	33
99.0 – 99.9		15	48
100.0 – 100.9		16	64
101.0 – 101.9		8	72
102.0 – 102.9		11	83
103.0 – 103.9		24	107
104.0 – 104.9		1	108

The last entry in the cumulative frequency column should be the total number of data items that we began with when we started the problem.

Section 4.1.5 Relative Frequency Distribution

A relative frequency distribution is a frequency distribution that shows the relative frequency of items in each of several non-overlapping classes. The relative frequency is the fraction or proportion of the total number of items belonging to a class and it is generally expressed as a percentage. This definition is applicable to both quantitative and qualitative data. You may wonder why we would need another distribution that is basically the same as a frequency distribution. The relative frequency distribution allows us to compare classes of data that may have been gotten from different sources or collected from different places. For example if we are comparing the number of crimes committed in Madison, Wisconsin, population 208,000, to Madison, Minnesota, population 1750, we would note that the Wisconsin Madison has much

more crime in all categories. However, when we look at the percentages, relative frequencies, for certain crimes they are about the same for both places.

To create a relative frequency distribution for a given data set. We first create the frequency distribution for the data. This distribution may be either grouped or ungrouped. We then take the frequency for each class and divide it by the total number of data set items. We should get a decimal answer which we can write to two decimal places. We list these decimals in a new column called relative frequency. The relative frequencies should all add up to 1.0; however, because we may have done some rounding we might be a little lower than one or a little above one.

EXAMPLE:

In the last example we created a grouped frequency distribution for the temperatures of 108 people who visited a clinic and had their temperature taken by a student nurse. What we would now like to do is to create a cumulative frequency distribution for that data set. We can use what we have created in that example by adding a new column to the end of the previous table called relative frequency. To determine the relative frequency for each class we take the frequency and divide it by the total. We do this for each class. For example the first class has a frequency of 17. We divide that by 108 to get 0.1574. We round to 0.16 and this is the relative frequency.

Temperature	Tally	Frequency	Relative Frequency
97.0 – 97.9		17	0.16
98.0 – 98.9		16	0.15
99.0 – 99.9		15	0.14
100.0 – 100.9		16	0.15
101.0 – 101.9		8	0.07
102.0 – 102.9		11	0.10
103.0 – 103.9		24	0.22
104.0 – 104.9		1	0.01

Your Turn!!

1. Covance was performing one of their medical studies and put out a call for health people who were between the ages of 18 and 40. The study would consist of a 5 day stay at their facility for which the participant would get \$800. The ages of the 107 people who applied to be in the study are given below.

22	23	22	19	18	20	27	19	19	24	35	25
25	35	21	33	19	24	18	22	30	20	29	18
31	28	27	21	30	27	32	33	23	29	25	31
25	28	30	33	28	22	21	30	22	22	20	19

33	31	26	18	23	28	19	21	27	29	38	29
22	22	26	32	25	21	25	24	25	35	23	31
28	29	26	32	32	21	20	24	21	28	27	28
29	19	24	29	25	27	25	21	19	27	32	24
32	19	18	27	34	25	18	32	31	26	23	

- Form an ungrouped frequency distribution for the ages given above
- Form a grouped frequency distribution for the ages above using 7 classes.
- Which distribution is more meaningful; that is, provides more or better information?

2. A local garbage contractor is interested in determining what the distribution of weights of garbage contained in one bag of garbage left at the curb for pickup. He randomly selects one crew to record the weight of the bags of trash before being tossed in the truck. The crew weighs 62 bags of garbage along its route.

10.8	20.0	27.6	38.1	27.9	21.9	21.8	49.3	33.3	35.5	44.4
45.2	33.1	10.4	44.4	24.1	37.6	39.0	17.2	31.6	23.1	27.6
31.0	4.1	23.2	43.0	51.9	20.2	36.8	19.0	15.4	20.7	17.0
45.4	49.4	18.1	17.4	23.7	19.1	21.0	21.2	29.1	15.0	4.4
24.8	20.8	25.8	14.6	28.4	27.2	54.5	13.1	45.8	34.8	37.3
51.6	24.8	26.2	11.3	28.2	11.1	11.7				

- Create a grouped frequency distribution with 8 classes.
- Create a cumulative frequency distribution

3. A 100gram bag of M & M plain candy contained 115 pieces of candy in the following colors: red (R), orange (O), yellow (Y), brown (N), blue (B), and green (G). The candies came out of the bag in the following order.

Y	O	G	G	R	Y	O	R	N	B	B	R
R	Y	N	R	B	N	Y	B	Y	Y	Y	Y
R	N	B	N	R	N	Y	O	R	R	G	B
Y	B	N	Y	O	B	R	R	Y	R	N	N
N	G	R	B	B	B	R	N	N	N	Y	N
Y	N	Y	B	B	O	N	G	N	Y	B	R
Y	G	N	N	B	O	N	N	R	R	R	R
O	N	R	N	G	Y	Y	N	O	G	N	Y
R	Y	N	N	R	N	N	Y	N	N	Y	N
N	O	N	Y	Y	Y	R					

- Construct a frequency distribution based upon the color of the candy.
- Construct a relative frequency distribution for the color of the candy.

4. In the last 76 years the academy award for best actress has be given to actresses of all ages. The ages of those actresses are given below.

22	37	28	63	32	26	31	27	27	28	30	26
29	24	38	25	29	41	30	35	35	33	29	38
54	24	25	46	41	28	40	39	29	27	31	38
29	25	35	60	43	35	34	34	27	37	42	41
36	32	41	33	31	74	33	50	38	61	21	41
26	80	42	29	33	35	45	49	39	34	26	25
33	35	35	28								

- Create a grouped frequency distribution of the data using 6 classes.
- Create a cumulative frequency distribution for the data.
- Create a relative frequency distribution for the data.

Section 4.2 Graphing

By now you are wondering why we are doing all this organizational stuff with the data. If you look back at the definition of statistics we have so far only collected and organized the data or information. The next step is to analyze and present the information to someone. We are going to switch the order of these last two items and talk about presentation next.

The most commonly used method used present data is by means of graph or a chart. Creating charts and graph is not difficult. Once we have organized the information, the construction of the visual display is fairly straightforward. The hard part is to decide upon the best method of presentation. Some of the methods available to us are bar charts, line charts, and circle charts.

Section 4.2.1 Bar Charts

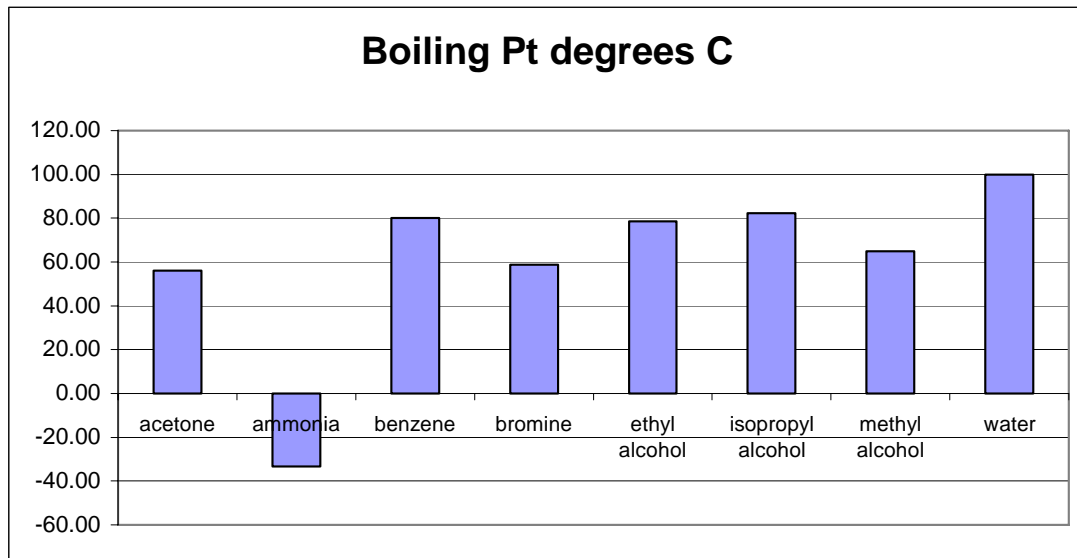
Every bar chart consists of two axes: a horizontal axis and a vertical axis. In standard practice, the horizontal axis is where we list the categories or classes of the data and the vertical axis is where we list the frequency, cumulative frequency, relative frequency or some other numerical value. There are two different kinds of bar charts: a bar chart and a histogram.

A bar chart is a graphical representation of data that is either categorical in nature or discrete, an ungrouped distribution. One of the easiest categorical bar chart to construct is one in which we have some quantity (where an amount is represented by the size of a numerical value) plotted against a category or quality or name. In general, the categories can be presented in any order. A common orderings is to list the categories alphabetically. Another common ordering is to list the categories based upon the ordering of the numerical value. However you decide to arrange the data just be consistent.

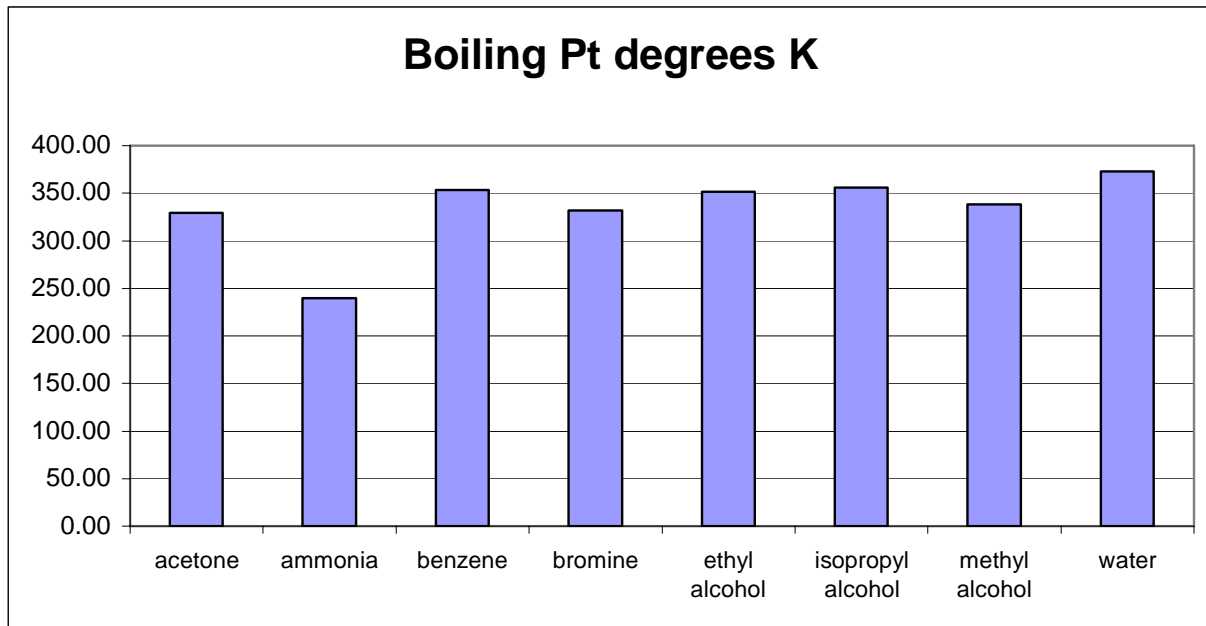
EXAMPLE:

The table below gives the boiling point at atmospheric pressure for eight different liquids. The boiling points are expressed both in degrees Celsius and in degrees Kelvin (i.e., absolute temperature). We can consider the information to be a categorical distribution where the name of the liquid is the category and the boiling temperature is the frequency.

Liquid	Boiling Pt degrees C	Boiling Pt degrees K
acetone	56.20	329.36
ammonia	-33.35	239.81
benzene	80.10	353.26
bromine	58.78	331.94
ethyl alcohol	78.50	351.66
isopropyl alcohol	82.40	355.56
methyl alcohol	64.96	338.12
water	100.00	373.16



We will use the Celsius data to construct the bar chart. On the horizontal axis we will list the names of the liquids and on the vertical axis we will list boiling temperatures in Celsius. Above each name we will construct a bar from the vertical axis up to the level where the temperature is located. We do this for each pair of data and the result is the above chart. If we had used the absolute temperatures we would have gotten a less exaggerated view of the same data. The absolute temperature chart is given below and is constructed the same way: the horizontal axis has the names of the liquids and the vertical axis lists boiling temperatures in absolute temperature.



Which chart is most appropriate depends upon what information we wish to emphasize. The differences between boiling points are easier to see on the first chart, while the second gives a more accurate comparison of the different liquids.

EXAMPLE:

Let say that what we were trying to determine how much blood of a specific type should be kept on hand so that if a transfusion were required the proper type of blood would be available. We will ignore some of the other factors that go into blood typing and just focus on the four major categories of A, B, AB and O. We type the blood of 50 randomly selected individuals and get the following results:

```

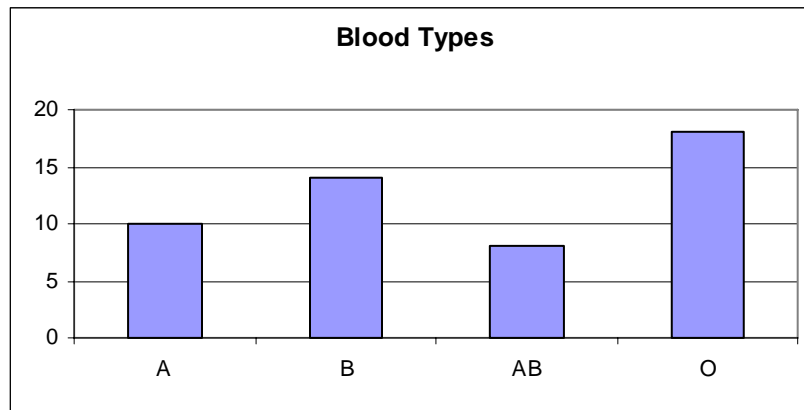
A   B   B   AB  O   O   O   B   AB  B
B   B   O   A   O   A   O   O   O   AB
AB  A   O   B   A   A   O   B   A   AB
B   O   B   O   A   B   B   O   O   O
AB  AB  A   O   B   O   B   O   AB  A

```

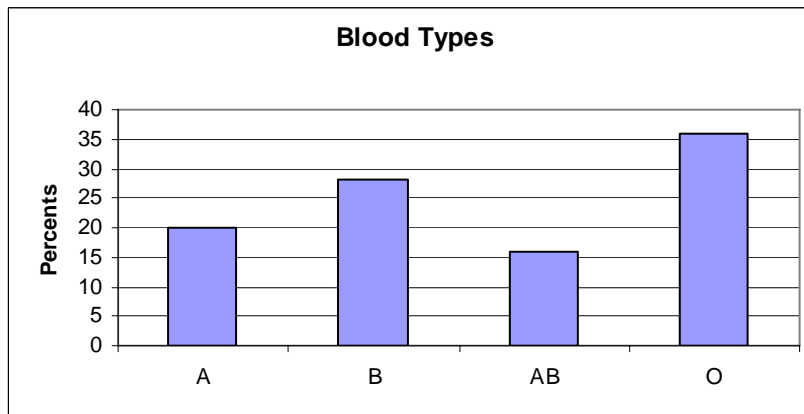
We would like to display our results using a bar chart. The first thing that we must do is to create a frequency distribution of the data. While we are at it, let us also create the relative frequency distribution for this information. The results are given below.

Type	Tally	Frequency	Relative Frequency
A		10	0.20
B		14	0.28
AB		8	0.16
O		18	0.36

Now that we have the results we can create the bar chart. We have four categories, one for each blood type so we will have four bars on the horizontal axis. We look at the frequencies and not they go as high as 18. So the upper limit of the frequency axis, the vertical, should be 20. Each bar should be the same width. We get the below graph.



If we had wanted to use the relative frequencies our graph would have looked very much like what we have above. The big difference is that instead of the frequency we would have relative frequency on the vertical axis as either percents or as decimals. In the below chart we use percents and tell the viewer that the frequencies are percents.



We can also use a bar chart to represent distributions of discrete numerical data. We do not want too many bars because if we have more than 7 or so the graph will begin to look cluttered. Also we do not want a lot of zero frequencies. If we do have a lot of zero frequencies we will have large gaps between the bars. Consider the following example of discrete numerical data being presented by means of a bar chart.

EXAMPLE:

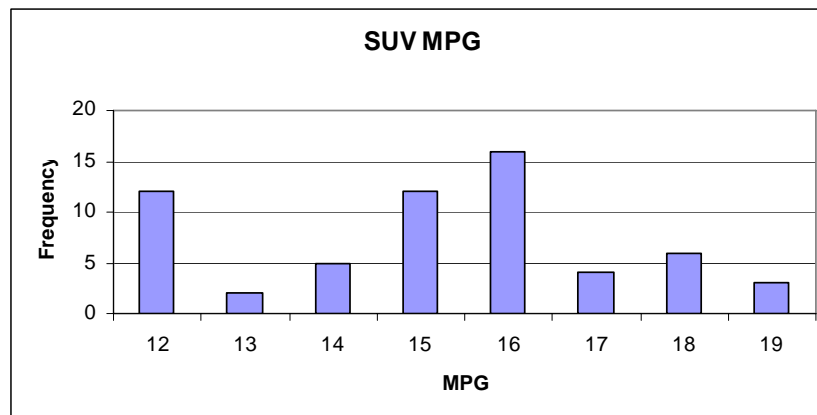
In 1999, sports utility vehicles averaged between 12 and 19 miles per gallon of gas. A survey was sent to 60 families who had sports utility vehicles and they were asked to compute the miles per gallon, rounded to the nearest mile, they got on their vehicle for a one month period. The results are given below.

12 17 12 14 16 18 16 18 12 16 17 15
 15 16 12 15 16 16 12 14 15 12 15 15
 19 13 16 18 16 14 12 16 15 12 19 17
 18 16 14 13 12 12 12 15 16 19 14 16
 15 12 18 16 17 16 15 16 18 15 16 15

To draw the graph we must first create the frequency distribution. It is given below.

MPG	Tally	Frequency
12		12
13		2
14		5
15		12
16		16
17		4
18		6
19		3

Now we can draw the bar chart because our categories are discrete numbers. The graph is given below.



Section 4.2.2 Histograms

A histogram is a type of bar chart in which the bars touch. It is used when we are trying to display continuous numerical data. The reason the bars touch is that one class of data transitions into the next class of data. The graph is drawn similarly to a bar chart: the horizontal axis represents the classes and the vertical axis is the frequency. On the horizontal axis we indicate either the midpoint of the class, under the bar, or we indicate the class limits at the edge of each bar beginning with the lowest class and going up to the end of the upper class.

Let's look at an example of a histogram,

EXAMPLE:

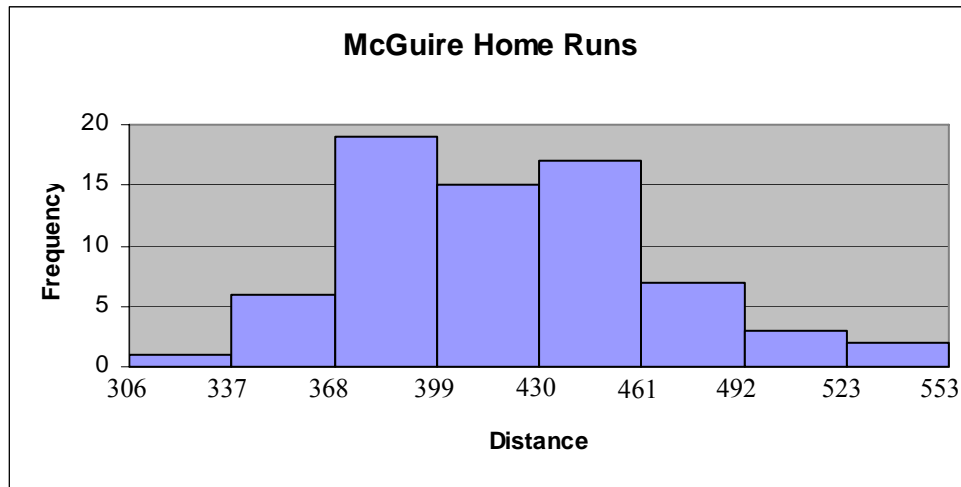
During the 1998 baseball season Mark McGuire hit 70 home runs. The distance in feet of each home run is given below. Create a histogram to display the data.

306	370	370	430	420	340	460	410	440	410	380	360
350	527	380	550	478	420	390	420	425	370	480	390
430	388	423	410	360	410	450	350	450	430	461	430
470	440	400	390	510	430	450	452	420	380	470	398
409	385	369	460	390	510	500	450	470	430	485	380
430	341	385	410	420	380	400	440	377	370		

To create the graph we must first create the frequency distribution. We will use 8 classes. The longest home run was 550 feet and the shortest was 306. The difference is 244 feet. We divide by 8 to get 30.1 which is rounded up to 31. The frequency distribution is as follows:

Distance	Tally	Frequency
306 – 336		1
337 – 367		6
368 – 398		19
399 – 429		15
430 – 460		17
461 – 491		7
492 – 522		3
523 – 553		2

The graph then is formed by bars that touch and the lower limits are listed at the left edge of each bar and the last upper limit is at the right of the last bar. The graph is shown below



Section 4.2.3 Line Charts

A line chart is any graph that is drawn using a set of points, consisting of two variables or data values, where each point is connected to the next point by means of a straight line. Each data value or variable represents a quantity: the first data value or variable represents a location on the horizontal axis and the second data value or variable is a vertical height or measure above the first data value or variable in a point.

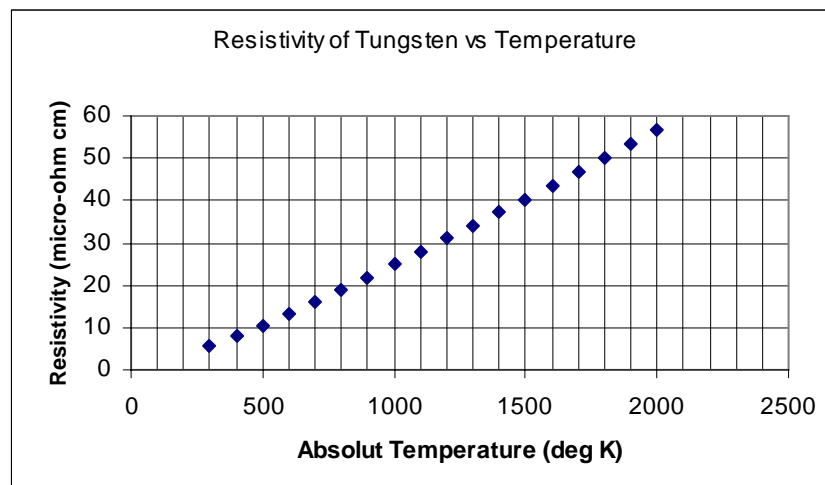
EXAMPLE:

Data on the electrical resistivity of tungsten versus absolute temperature is given below.

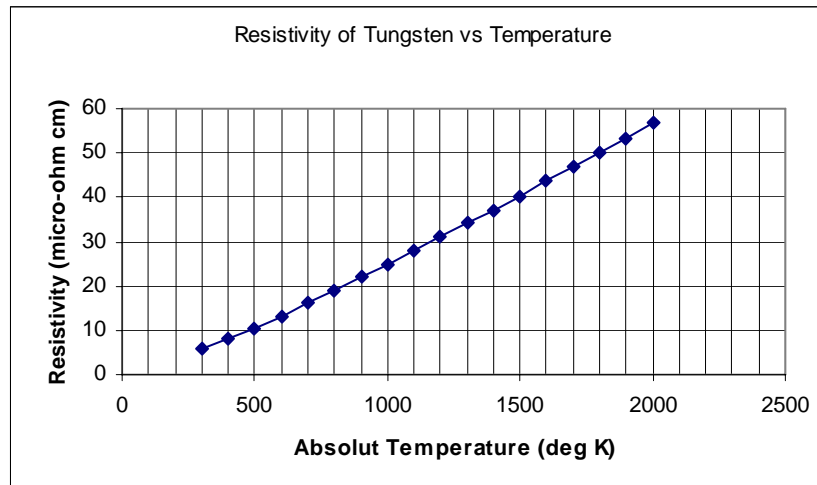
Temp Degree K	Resistivity micro ohm cm
300	5.65
400	8.06
500	10.56
600	13.23
700	16.09
800	19.00
900	21.94
1000	24.93
1100	27.94
1200	30.98
1300	34.08
1400	37.19
1500	40.36
1600	43.55
1700	46.78
1800	50.05
1900	53.35
2000	56.67

Each data pair from the table could be represented as a point on a graph. The first data value of the point, Temperature, would be located on the horizontal axis and the second data value of a point, Resistivity, would be the vertical height of the point on the graph.

From this table we construct the following graph by using a two dimensional grid. The horizontal axis is often called the x -axis, while the vertical axis is the y -axis. The two axes intersect at the point $(0, 0)$, often called the origin. To graph the data we first determine which variable is " x ", and which is " y ". Each data point is plotted by moving horizontally from the origin by an amount x and vertically an amount y . If x is positive we move to the right and if x is negative we move left from the origin, while if y is positive we move upwards and if y is negative we move downwards from the origin. The graph of the above data is the points plotted below.



We can see from the graph that the data points look like they will form a line. When this occurs we say that the relationship between the two variables is "approximately linear". If we connect the points with a series of straight lines, we can use the graph to estimate values of the resistivity that were never measured. The graph below shows the plotted point with the addition of the connecting lines.



We can use this graph to estimate resistance for temperatures that were not measured. For example, when the temperature is 1050 degrees K, we can estimate a value of about 27 micro ohm cm for the resistivity of tungsten. We do this by determining where a temperature of 1050° K would be on the horizontal axis and then go straight up until we get to the line. At that point we go to the vertical axis and approximate the resistance. In a similar manner we could estimate the temperature required for a given level of resistance. If we wanted to know the temperature at which we would have a resistance of 15 micro-ohm cm we go up the vertical axis until we get to approximately 15. We then go to the line on our graph and at that point we go straight down to the horizontal axis to read about 650° K.

Section 4.2.4 Cumulative Line Charts

Earlier we looked at how to create a cumulative frequency distribution. We did this by first creating a frequency distribution and then for each class adding together the frequencies for all class up to and including that class. We can draw a graph of the accumulated data by means of a line chart. This chart is drawn just like the chart in the previous example. The classes are listed along the horizontal axis and the accumulated frequencies are plotted vertically above each class. The points are then connected with lines. If our classes represent grouped data the point is plotted at the upper limit.

EXAMPLE:

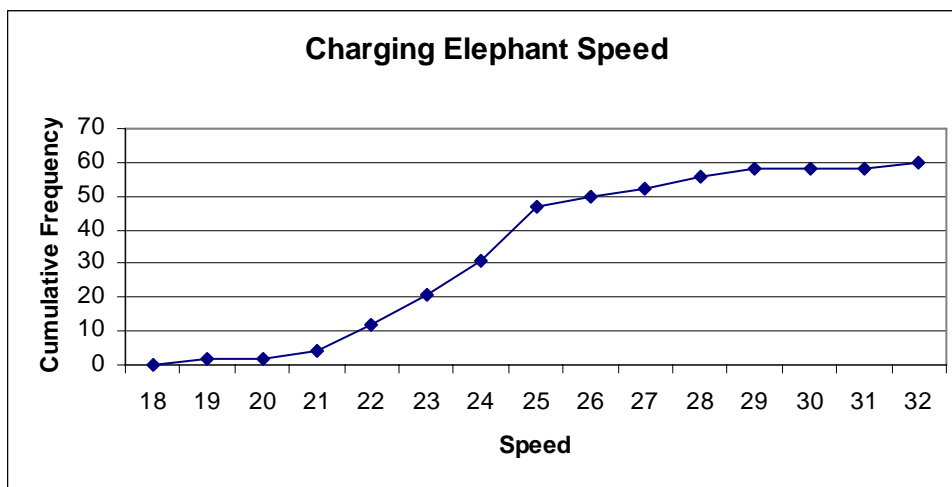
A veterinarian technician reads in the *World Almanac and Book of Facts* that a charging elephant can attain speeds of up to 29 miles per hour. He decides to see if this is true. He saves his money, buys a used radar speed gun, and goes on safari to Africa. While on safari he gets 60 elephants to charge the vehicle in which he has his equipment. The speeds are recorded below. Create an accumulative frequency distribution and draw the graph.

25 24 25 24 25 23 25 19 32 23 22 24
 22 25 32 28 25 25 26 27 22 28 24 23
 24 21 22 25 22 29 23 25 23 22 28 22
 21 24 25 24 25 28 25 25 19 26 25 24
 22 24 23 25 26 23 29 25 23 23 27 24

The distribution is as follows:

Speed	Tally	Frequency	Cumulative frequency
19		2	2
20		0	2
21		2	4
22		8	12
23		9	21
24		10	31
25		16	47
26		3	50
27		2	52
28		4	56
29		2	58
30		0	58
31		0	58
32		2	60

The graph of the cumulative distribution is given below.

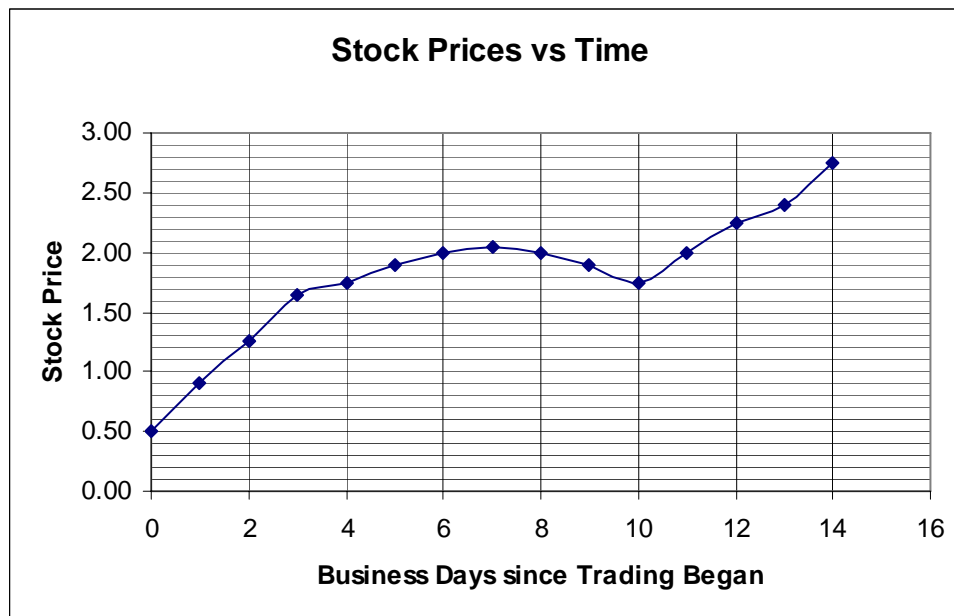


Section 4.2.5 Time-Series Line Charts

A time series graph is a graph that plots data over time. The horizontal axis is the time axis. It can be expressed in any interval of time: minutes, hours, days, months, years, and etcetera. The vertical axis is the frequency or value. The points relate the frequency or value to a specific time.

EXAMPLE:

We will plot stock market prices versus time for a particular stock. The stock is increasing in value when the curve moves up going from left to right and is decreasing when the curve moves down from left to right. For example, in the graph shown below, stock prices are increasing on the days in the interval 0 to 7 and the days in the interval 10 to 14. Stock prices are decreasing on the days in the interval 7 to 10. The stock prices "peaked" or had a "local maximum" at day 7, when the price was \$2.05. The stock prices "bottomed out" or had a "local minimum" at day 10, when the price was \$1.75. A local maximum always happens at a transition from increasing to decreasing. Similarly, a local minimum occurs at a transition from decreasing to increasing.



Section 4.2.6 Equation Line Charts

Graph can also be generated from equations. An equation is a relation between two variables x and y . In an equation, we usually imagine that x is the independent or input variable (whose value we control or set), and that y is the dependent or output variable (what we measure or record). When the curve has y "going up" as x moves from left to right, we say that y is

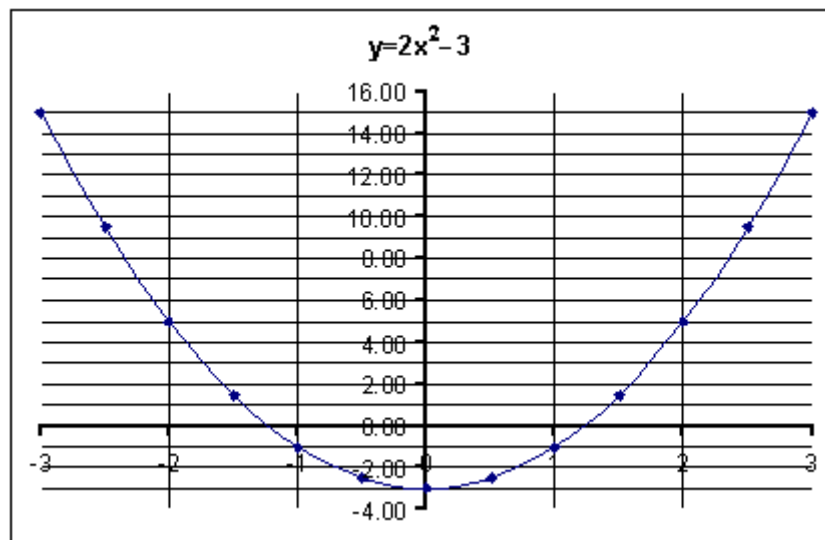
"increasing", while if the value of y goes down as x moves from left to right, y is "decreasing". This follows the same reasoning that we saw in the time-series graph and discussion above.

EXAMPLE:

Suppose we have the formula, $y = 2x^2 - 3$, if we substitute, "plug-in," various values of x into this formula results in the following table.

x	$y = 2x^2 - 3$
-3.00	15.00
-2.50	9.50
-2.00	5.00
-1.50	1.50
-1.00	-1.00
-0.50	-2.50
0.00	-3.00
0.50	-2.50
1.00	-1.00
1.50	1.50
2.00	5.00
2.50	9.50
3.00	15.00

Using this table we construct the graph shown below.



Notice that our graph results in a series of points. We could have connected each pair of consecutive points with a straight line but it made more sense to connect them with a curved segment so that the graph is smooth. A graph of this type is called a curve. The graph has a local minimum when $x = 0.00$.

Section 4.2.7 Circle Graphs

A circle graph which is sometimes called a pie graph is a circle that is divided into sections or wedges according to the percentages of frequencies in each category of the distribution. We can use pie graphs to help us represent the data in a relative frequency distribution. Pie graphs are used most often with categorical or attributive data.

To construct a pie graph we first must create the relative frequency distribution of percentages of data within each category or class. Then we take each percentage and multiply by 360° to determine about how large each wedge or section of the pie is going to be. We then draw the wedge and label it.

EXAMPLE:

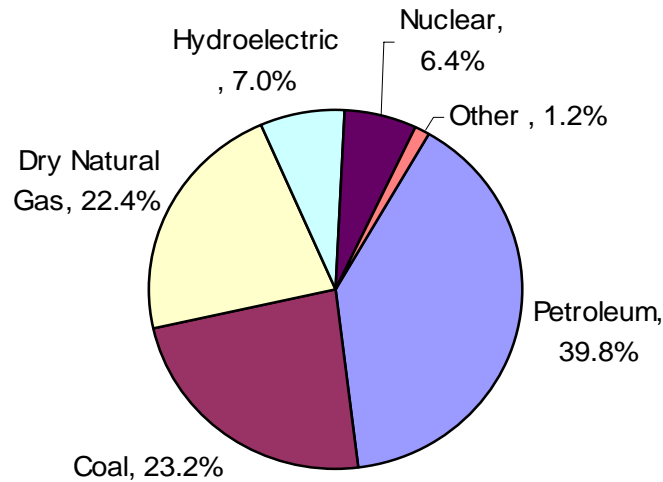
The N. Y. Times Almanac in 2002 listed the following percentages of world wide energy use from different sources. Construct a pie graph of the relative frequency distribution.

<u>Energy Type</u>	<u>Percentage used</u>
Petroleum	39.8%
Coal	23.2%
Dry Natural Gas	22.4%
Hydroelectric	7.0%
Nuclear	6.4%
Other (Wind, Solar, etc)	1.2%

We first change the percentages to decimal and multiply by 360o to determine the size of each wedge. Then we can draw the graph.

<u>Energy Type</u>	<u>Percentage used</u>	<u>Degrees</u>
Petroleum	39.8%	143
Coal	23.2%	84
Dry Natural Gas	22.4%	81
Hydroelectric	7.0%	25
Nuclear	6.4%	23
Other (Wind, Solar, etc)	1.2%	4

Energy Usage by Type



Your Turn!!

1. Below is a table which gives the electrical resistivity (good conductors have less resistivity) in micro-ohm-cm of various metallic elements designated by their chemical symbol.

Element	Al	Cu	Au	Pb	Ni	Pt	Ag	Zn
Resistivity	2.65	1.67	2.35	20.64	6.84	10.6	1.59	5.92

- Construct a bar graph that displays this information.
 - Which metal is the best conductor?
 - Which metal is the poorest conductor?
2. A sporting goods store collected data about what type of balls its customers bought over the course of one weekend day's worth of selling. Those little cards do provide the store with all sorts of information. In the list below B is a baseball, K is a basketball, F is a football, G is a package of golf balls, S is a soccer ball, and T is a container of tennis balls.

F G F F K B K G T S B F
 T S G B K G B S T K G G
 G S S T T G K B F K F B
 B B G T F K G G F S K G
 T F T T T S K T F S S G

- Construct a bar chart depicting the number of balls sold
- Construct a pie chart displaying the percentages of balls sold.

3. The Federal Highway Administration collected data to determine what the added cost to operate a vehicle was due to bad road conditions. They interviewed 40 people about their recent repair work and determined that part of that cost could be attributed to the conditions of the roads over which the people drove. The estimated cost added to the vehicle for these 40 people is given below.

165	156	136	116	91	168	114	127	90	170
122	113	112	187	111	172	135	188	136	147
140	131	171	159	165	153	125	179	97	163
208	177	152	141	159	169	136	155	85	150

- Construct a histogram, bar chart in which the bars touch, for this data. Use 6 classes.
- Construct a cumulative line chart for this data.

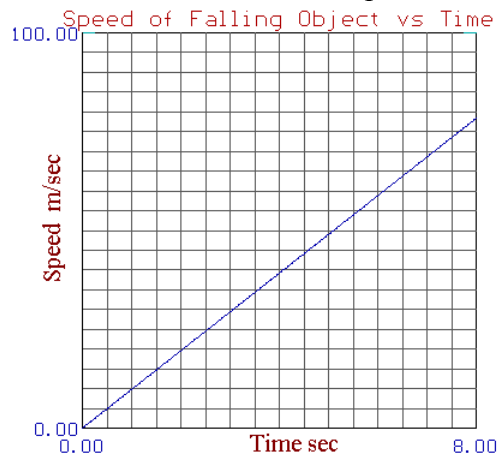
4. The data below represents the federal minimum hourly wage for the years shown.

1960	1965	1970	1975	1980	1985	1990	1995	2000	2005
\$1.00	\$1.25	\$1.60	\$2.10	\$3.10	\$3.35	\$3.80	\$4.25	\$5.15	\$5.15

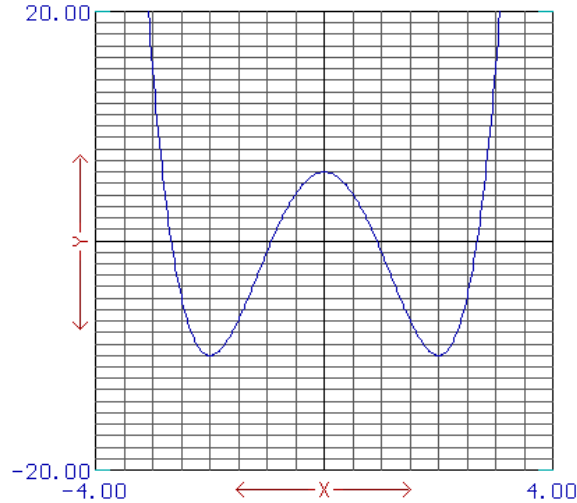
- Draw a times series chart for the federal minimum wage.
- In what years did the graph have a maximum value for the federal minimum wage?
- In what 5 year period did the graph increase the most?

5. The following graph gives the speed of a falling object in m/sec as a function of time measured in sec.

- What is the speed when the time is 3.0 sec?
- At what time is the speed 50m/sec?
- If the speed V is related to the time t by the equation: $V = gt$, where g is the acceleration of gravity. Estimate the value of g from this graph.
- Using the value of g just calculated; estimate the speed when the time is 10.0 sec.

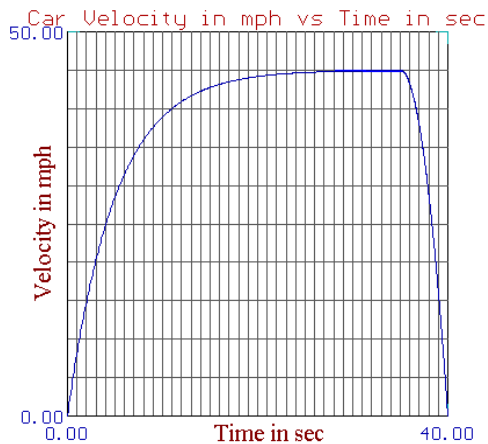


6. Below is shown the graph of y versus x . Answer the following questions.



- When $x = -1$, estimate the value of y .
- When $x = 0$, estimate the value of y .
- When $x = -3$, estimate the value of y .
- When $x = 2$, estimate the value of y .
- When $x = 2.5$, estimate the value of y .
- For what intervals of x values is y increasing?
- For what intervals of x values is y decreasing?
- For what values of x does y have a local minimum?
- For what values of x does y have a local maximum?

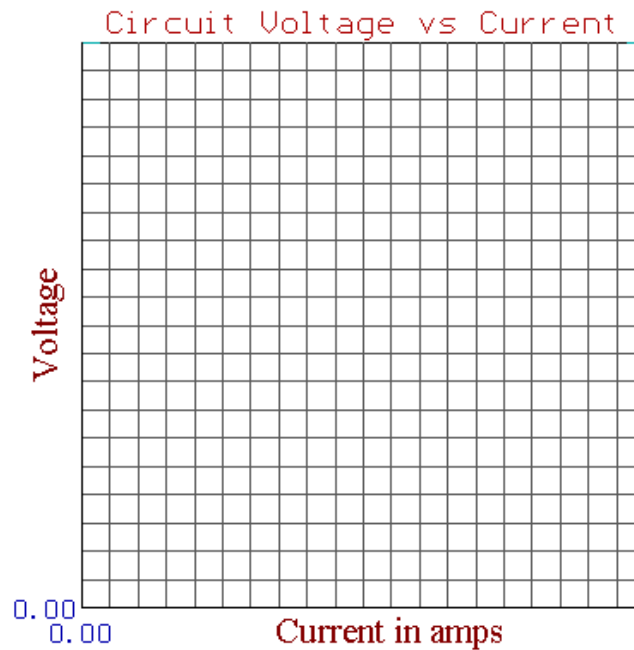
7. Below is shown the graph of velocity (positive means motion to the right, negative means motion to the left) versus time of a car travelling between two stop lights. The velocity is in units of mph and the time is in units of seconds.



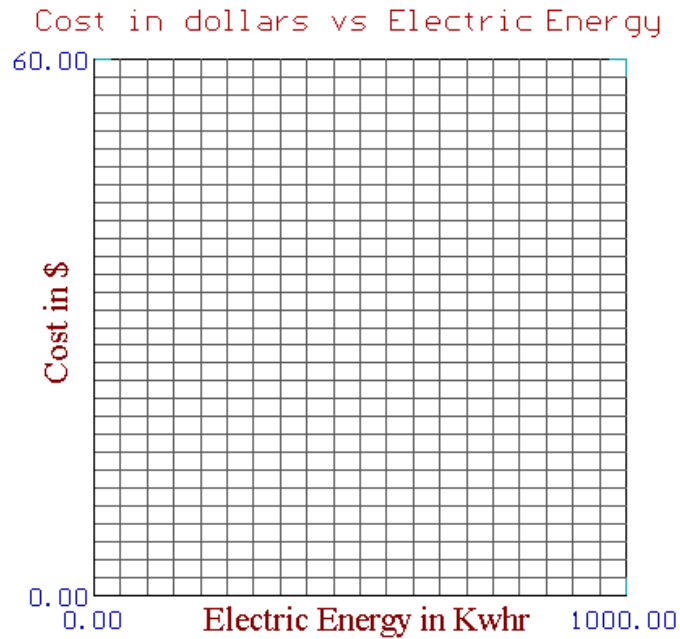
- a. How fast was the car moving at the first stop light?
 - b. How fast was the car moving at the second stop light?
 - c. Estimate the car’s greatest speed.
 - d. How long after it left the first stop light did the car travel to the right?
8. For a given circuit the amount of applied voltage, V , and the current in amps, I , are related by means of the follows table.

V Voltage	4.0	8.0	12.0	16.0	20.0	24.0	28.0	32.0	36.0
I Current	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5

- a. Construct a graph of Voltage versus Current from the following table:
- b. From your graph estimate V when I is 5.0 amps.



9. The cost of electric power is given by the equation $C = \$0.0558E$, where E is the amount of electrical energy consumed in kilowatt hours (Kwhr).
- a. Generate a graph of C versus E , for $0 < E < 1000$ Kwhr.
 - b. Estimate the value of C when E is 600 Kwhr.



Section 4.3 Descriptive Statistics

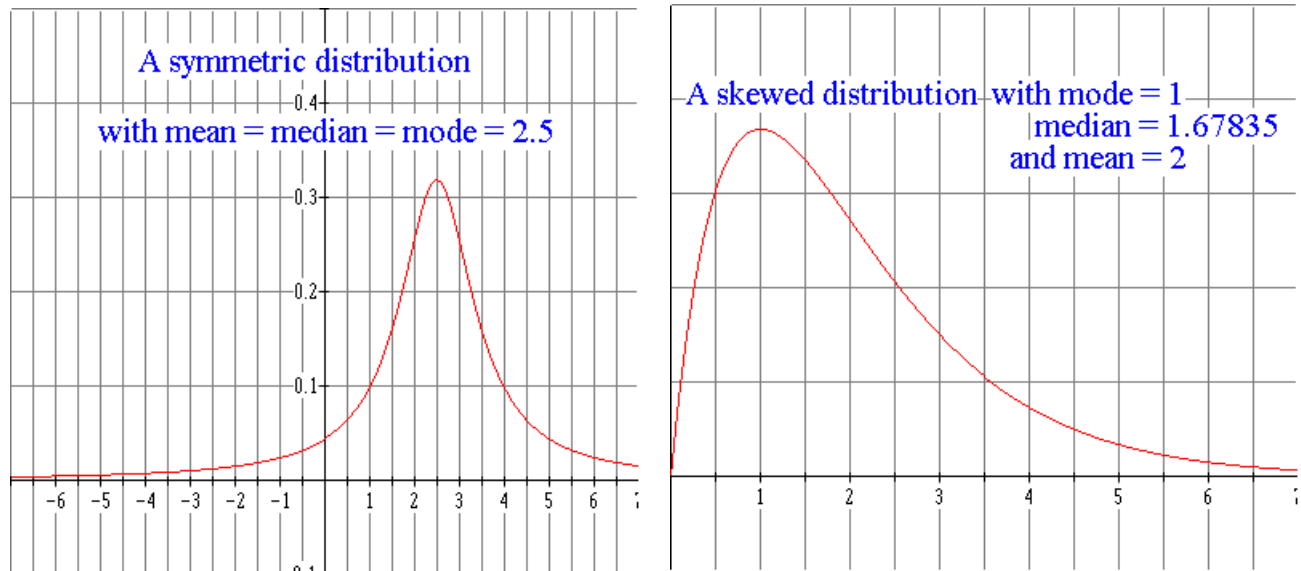
Descriptive statistics is the body of methods used to represent and summarize sets of numerical data. Descriptive statistics provides a means of describing how a set of measurements (for example, people's weights measured in pounds) is distributed. Descriptive statistics involves two different kinds of summary statistics. These summary statistics are called measures of center and measures of spread.

Section 4.3.1 Measures of Center

Measures of center involve a "typical" data value about which all the other data values are distributed. The most commonly used measures of center are the mean, the median and the mode. The mean is sometimes called the arithmetic average and it is the sum of all the data values divided by the number of data values. If the data is arranged in order, high to low or low to high, then the median is the "middle" data value with as many data values above it as below it. The mode is the data value that occurs the most often (has the highest frequency). The mode is only meaningful for large sets of data where it is likely for the same score to be measured more than once.

If we were to graph a large data set and the mean, median, and the mode were all the same, our graph would look like the graph below on the left. A graph of this type is called symmetric: one side of the graph looks just like the other side of the graph. Symmetric graphs result from distributions of data in which the mean, median and mode are all equal. If any of these three are

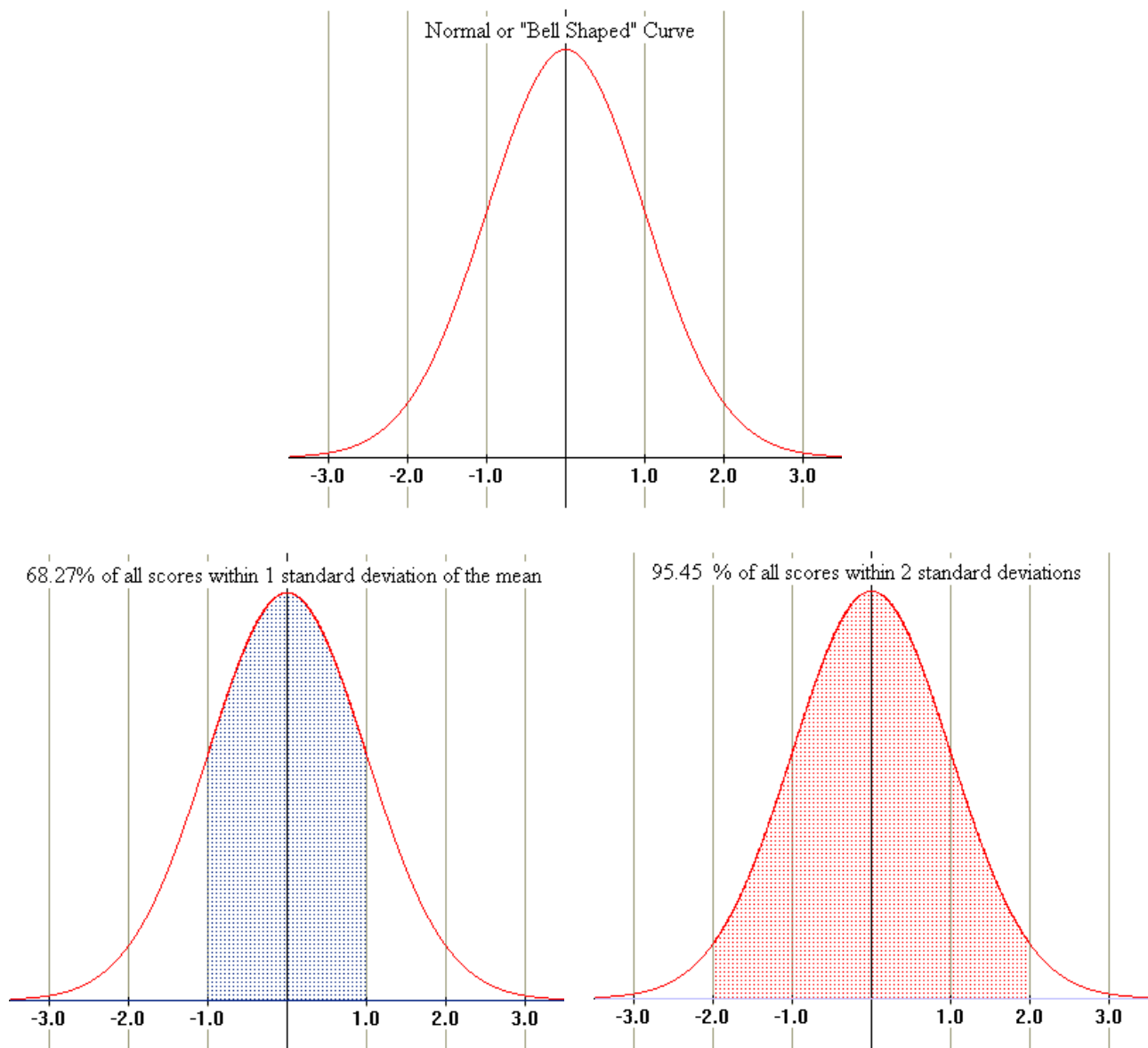
different from the other two or if all three are different the graph will no longer be symmetric and will become skewed. Therefore, any differences between these three measures are an indication of skewness or departure from a symmetric distribution.



Section 4.3.2 Measures of Spread

The most commonly used measures of spread are range, standard deviation, and inter-quartile range. The range is the distance between the highest data value (the maximum) and the lowest data value (the minimum). The standard deviation measures a “typical” distance of data values in the distribution from the mean. The inter-quartile range is the distance covered by the “middle half” of the data from the score that marks the first quarter point to the score that marks the third quarter point.

For the famous “bell-shaped” curve (what is generally called a normal distribution curve or a Gaussian distribution), 68% of all scores are found within one standard deviation of the mean and 95.5% of all scores are within two standard deviations of the mean. More generally, for any distribution of scores, **at least** 75% of all scores must be within two standard deviations of the mean.



Section 4.3.3 Calculation Notation

To make the calculation of these statistics easier the following notation will be used.

n represents the number of data values which make up the data set or the sample

x_i represents a subscripted variable. The index i can take values from 1 to n with each value of x_i being a particular data value in the data set. Furthermore, it will be assumed that the scores have been sorted from smallest to largest (an increasing sort). Thus, x_1 is the minimum and x_n is the maximum.

f_i represents the frequency for a corresponding x_i

Σ is the upper case Greek letter sigma. This symbol stands for the mathematical operation of summation and means we take the sum of the data values: $x_1 + x_2 + x_3 + \dots + x_n$. This sum can

be more concisely written as $\sum_{i=1}^n x_i$. The variable i is called a “dummy” summation index and

is just used to signify that various values of x are being summed. The designation $i = 1$ beneath the sigma indicates where the sum begins and the n above the sigma indicates where the sum ends. In statistics it is nearly always the case that sums begin at 1 and end at n , so the following abbreviated symbols are often employed.

$$\sum_{i=1}^n x_i = x_1 + x_2 + x_3 + \dots + x_n = \sum_i x_i = \sum x_i = \sum x .$$

Section 4.3.4 Mean Calculations

The mean is the arithmetic average and it is usually indicated by \bar{x} (called x bar). Sometimes a second symbol for the mean is used, called the population mean, when all possible data values have been measured. The population mean is represented by the Greek letter lower case mu, μ .

The mean can be computed by adding together all of the data values and dividing by the number of data values that were added together. In terms of a formula, we write the mean calculation as

$$\bar{x} = \frac{\sum x_i}{n} .$$

EXAMPLE:

In the years 1992 – 1994 the United States had 18 space shuttle missions. The duration of these missions in days is given below. Compute the mean of the space shuttle missions.

8	9	9	14	8	8	10	7	6
9	7	8	10	14	11	8	14	11

$$\bar{x} = \frac{\sum x_i}{n}$$

$$\bar{x} = \frac{8+9+9+14+8+8+10+7+6+9+7+8+10+14+11+8+14+11}{18}$$

$$\bar{x} = \frac{171}{18}$$

$$\bar{x} = 9.5$$

If there is a lot of data, we may want to form a frequency distribution before we begin to do any computations. The benefits are that the computations we will have to do are reduced because we rely on the fact that multiplication is repeated addition. We do have another formula for the mean that we will use when we have a frequency distribution.

$$\bar{x} = \frac{\sum x_i \cdot f_i}{\sum f_i}$$

If the data is ungrouped we use each data value for x_i in the formula. If the data is grouped then we have to determine the midpoint of each class and use that for x_i . The midpoint is the average of the upper and lower limits of the class

EXAMPLE:

A 15 question practice driver's education test was administered to a group of people planning on taking the written portion of the driver's license examination. The number of correct questions a person got is shown below.

Correct	Frequency
3	3
4	4
5	7
6	10
7	13
8	14
9	16
10	14
11	9
12	10
13	8
14	7
15	5

Compute the mean number correct.

We note that in the formula $\bar{x} = \frac{\sum x_i \cdot f_i}{\sum f_i}$ that the numerator requires we sum the product of the

x 's times their corresponding f 's. To do this we will create a new column to record the products. The denominator requires us to sum all the frequencies so that we can divide the two sums.

Our new table looks like

Correct	Frequency	
x	f	$x \cdot f$
3	3	9
4	4	16
5	7	35
6	10	60
7	13	91
8	14	112
9	16	144
10	14	140
11	9	99
12	10	120
13	8	104
14	7	98
15	5	75
Total	$\sum f_i = 120$	$\sum x_i f_i = 1103$

We now substitute into the formula and simplify.

$$\bar{x} = \frac{\sum x_i \cdot f_i}{\sum f_i} = \frac{1103}{120} = 9.19$$

The mean number correct on the test was 9.19 correct.

EXAMPLE:

A copy center has 75 copy machines. At some point each machine required some type of service. The number of days between servicing for the machines is given below in the table. Compute the mean days between required servicing of the machines.

Days	Frequency
15.5 – 18.5	14
18.5 – 21.5	12
21.5 – 24.8	18
24.5 – 27.5	10
27.5 – 30.5	15
35.5 – 33.5	6

We must first compute the midpoint of each class. This is done by computing the midpoint of the first class, add the limits and divide by 2, and then add the class to the midpoint to get the next midpoint. Keep adding the width to get each following midpoint.

Days	Midpoint	Frequency	
	\bar{X}	f	$x \cdot f$
15.5 – 18.5	17	14	238
18.5 – 21.5	20	12	240
21.5 – 24.8	23	18	414
24.5 – 27.5	26	10	260
27.5 – 30.5	29	15	435
35.5 – 33.5	32	6	192
Total		$\sum f_i = 75$	$\sum x_i f_i = 1779$

We now substitute into the formula and simplify.

$$\bar{x} = \frac{\sum x_i \cdot f_i}{\sum f_i} = \frac{1779}{75} = 23.72$$

The mean number of days between servicing is 23.72 days.

Section 4.3.5 Mode Calculations

The mode of a data set is the data value that occurs the most often in the data set. If we have a frequency distribution, the mode is the data value with the largest frequency. If the frequency distribution is a grouped distribution, then we can't really determine which data value occurs most but we can determine which class occurs most often.

Example:

In the years 1992 – 1994 the United States had 18 space shuttle missions. The duration of these missions in days is given below. Determine the mode of the space shuttle missions.

8 9 9 14 8 8 10 7 6
9 7 8 10 14 11 8 14 11

We will form a frequency distribution for the data. The value that occurs most often is the mode. From the table we see that 8 days occurs most often. That is the mode of this data.

Days	Tally	Frequency
6		1
7		2
8		5
9		3
10		2
11		2
14		3

Example:

A copy center has 75 copy machines. At some point each machine required some type of service. The number of days between servicing for the machines is given below in the table. Compute the mode of the days between required service for the machines.

Days	Frequency
15.5 – 18.5	14
18.5 – 21.5	12
21.5 – 24.8	18
24.5 – 27.5	10
27.5 – 30.5	15
35.5 – 33.5	6

We look at the distribution and see that the class 21.5 – 24.8 has the highest frequency. This is the modal class. Most repairs will occur between 21.5 days and 24.5 days.

Section 4.3.6 Median Calculations

The median of a set of data values is the data value or average of two values that is located in the exact middle of the data set. The median can be calculated by the following procedure.

- Arrange all of the data values in order from low to high.
- Determine how many data values, n , are in the data set.
- Use the formula $d(m) = 1 + \frac{1}{2}(n - 1)$ to determine the position or depth of the median in the data set.
 - If $d(m)$ is a whole number.

Beginning at the low end of the data, count each data value until you get to $d(m)$. This is the median. There should be as many data values below the median as above the median.

- ii. If $d(m)$ is a decimal then $d(m)$ will be located between two data values.
 Beginning at the low end of the data, count each data value until you get to the whole number part of the decimal $d(m)$.
 Multiply the difference of the two number either side of where $d(m)$ fell by the decimal part of $d(m)$. This will always be 0.5.
 Add this to the number below $d(m)$. The result is the median. There should be as many data values below the median as above the median.

Example:

It has been observed that time seems to drag on in some classes. This is not one of those classes. To determine if people could really determine when one minute (60 seconds) has passed 13 students separated from other students were asked to indicate when they thought one minute had elapsed. The recorded times for these students is given below. What is the median time from this sample of students?

53 52 75 62 68 58 49 49 65 63 51 64 54

To determine the median we order the data from low to high.

49 49 51 52 53 54 58 62 63 64 65 68 75

There are 13 data values so $d(m) = 1 + \frac{1}{2}(n-1) = 1 + \frac{1}{2}(13-1) = 1 + \frac{1}{2}(12) = 1 + 6 = 7$. The median is in the 7th position.

49 49 51 52 53 54 58 62 63 64 65 68 75

The median is 58 as this is in the 7th position of the arranged data.

Example:

Listed below are the intervals in minutes between eruptions of the Old Faithful geyser in Yellowstone National Park. What is the median time between eruptions?

98 92 95 87 96 90 65 92 95 93 98 94
 To determine the median we order the data from low to high.

65 87 90 92 92 93 94 95 95 96 98 98

There are 12 data values. We compute

$$d(m) = 1 + \frac{1}{2}(n-1) = 1 + \frac{1}{2}(12-1) = 1 + \frac{1}{2}(11) = 1 + 5.5 = 6.5.$$

The median is between the 6th and 7th data values.

65 87 90 92 92 93 94 95 95 96 98 98

The difference between 94 and 93 is 1.

The product of 0.5 and 1 is 0.5.

Add 0.5 to the number in the position below the depth of the median which is 93. The sum is 93.5 which is the median. The median time between eruptions is 93.5 minutes.

Medians can also be determined for data that is given to us as a frequency distribution. Most of the process of determining the median will be the same as what we did earlier. The data will already be ordered. We must compute how many data values there are; we do this by summing

the frequencies. We compute the depth of the median, $d(m) = 1 + \frac{1}{2}(\sum f_i - 1)$. Notice that n

was replaced by $\sum f_i$ but it is still how many data values there are in the data set. We must

form the cumulative distribution for the data. Compare the depth of the median with the cumulative frequencies beginning with the smallest cumulative frequency. The median class is the first class that has a cumulative frequency larger than the depth of the median.

EXAMPLE:

A 15 question practice driver's education test was administered to a group of people planning on taking the written portion of the driver's license examination. The number of correct questions a person got is shown below. Determine the median number correct.

Correct	Frequency
3	3
4	4
5	7
6	10
7	13
8	14
9	16
10	14
11	9
12	10
13	8
14	7
15	5

The data is already ordered so we must create the cumulative frequency distribution. As we are doing this we will also determine $\sum f_i$ because that should be the last entry in the cumulative column.

Correct	Frequency	Cumulative Frequency
3	3	3
4	4	7
5	7	14
6	10	24
7	13	37
8	14	51
9	16	67
10	14	81
11	9	90
12	10	100
13	8	108
14	7	115
15	5	120

We now compute the depth of the median.

$$d(m) = 1 + \frac{1}{2}(\sum f_i - 1) = 1 + \frac{1}{2}(120 - 1) = 1 + \frac{1}{2}(119) = 1 + 59.5 = 60.5.$$

The first class that exceeds the depth of the median is the 9 class with 67. The previous classes only totaled up to 51. The median number correct is 9.

Section 4.3.7 Range Calculations

The range, symbolized by an R , is a measure of the spread of the data. The range is the highest data value (H) minus the lowest data value (L). That is, $R = H - L$. The range of the data can be affected by extreme values.

EXAMPLE:

Listed below are the intervals in minutes between eruptions of the Old Faithful geyser in Yellowstone National Park. What is the range of times between eruptions?

98 92 95 87 96 90 65 92 95 93 98 94

The highest value is 98 and the lowest value is 65. The range is computed as follows:
 $R = H - L = 98 - 65 = 33$. The range of times is 33 minutes.

Section 4.3.8 Standard Deviation Calculations

Standard deviation is method statistical tool that allows us to determine how much variability is in the data. If we have a large standard deviation then there is a lot of variability. If there is a small standard deviation then there is little variability. Most manufacturing processes would like to keep variability to a minimum. Consider a group of soft drinks sold in 2-liter bottles on a shelf in the grocery store. Some of them look fuller than the others. This difference is a result of variability which we can measure by means of the standard deviation.

Standard deviation is calculated by the following formula $s = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n-1}}$. In the formula we use the symbol s to indicate the sample standard deviation. If all possible data values have been measured, then we have the slightly different formula (which gives a slightly smaller result)

because we can compute the population standard deviation $\sigma = \sqrt{\frac{\sum (x_i - \mu)^2}{n}}$. The Greek letter σ is the lower case sigma. It is used to designate that this is for a population of data values as opposed to just a sample. Both of these formulas can be manipulated using algebra to give the computationally more efficient formulas shown below.

For a sample standard deviation we would rather use the formula $s = \sqrt{\frac{\sum x_i^2 - \frac{(\sum x_i)^2}{n}}{n-1}}$ and for

a population standard deviation the formula $\sigma = \sqrt{\frac{\sum x_i^2 - \frac{(\sum x_i)^2}{n}}{n}}$.

The order of operations is crucial. $\sum x_i^2$ means that each x , a data value, is first squared and then summed (i.e., this is the sum of squares). In contrast, $(\sum x_i)^2$ means that the x 's are first summed and then this answer is then squared (i.e., this is the square of the sum).

EXAMPLE:

We have a data set that has 5 data values which are 1, 3, 4, 5, and 7. Then $n = 5$ and $x_1 = 1, x_2 = 3, x_3 = 4, x_4 = 5,$ and $x_5 = 7$ and

$$\sum x_i^2 = 1 + 9 + 16 + 25 + 49 = 100, \text{ while}$$

$$(\sum x_i)^2 = (1 + 3 + 4 + 5 + 7)^2 = 20^2 = 400.$$

If we compute the sample mean of this data set we have $\bar{x} = \frac{20}{5} = 4$, and if we compute the

sample standard deviation we have $s = \sqrt{\frac{100 - \frac{400}{5}}{5-1}} = \sqrt{5} = 2.2361$

If we treat this data as population data then the results are that the population mean is

$\mu = \frac{20}{5} = 4$, and the population standard deviation is $\sigma = \sqrt{\frac{100 - \frac{400}{5}}{5}} = \sqrt{4} = 2$.

Note that the means are the same but the standard deviations are a little different.

If our data is presented to us as a frequency distribution, the formulas will change a little because we now have frequencies to deal with.

For a sample the standard deviation formula becomes $s = \sqrt{\frac{\sum f_i x_i^2 - \frac{(\sum f_i x_i)^2}{\sum f_i}}{(\sum f_i) - 1}}$ and the

population standard deviation becomes $\sigma = \sqrt{\frac{\sum f_i x_i^2 - \frac{(\sum f_i x_i)^2}{\sum f_i}}{\sum f_i}}$. If the data is grouped then

the x_i 's are the midpoints of the classes.

These formulas may seem extremely complicated but if we arrange the data in a table the computations become easier. Let's look at a process that should help to make this manageable.

For sample data that we get in a list,

1. List the data in the first column
2. Determine the sum of the first column. This is $\sum x_i$
3. In the second column list the squares of the data in the first column
4. Determine the sum of the second column. $\sum x_i^2$
5. Determine how many data values there are. This is n .
6. Substitute into the formula for sample or population and simplify.

EXAMPLE:

In January 10 cities were selected at random and their high temperatures were recorded. The sample is given below. Compute the mean and standard deviation for this data.

50 37 29 54 30 61 47 38 34 61

To determine the sample mean we will use the formula $\bar{x} = \frac{\sum x_i}{n}$ and for the sample standard

deviation the formula $s = \sqrt{\frac{\sum x_i^2 - \frac{(\sum x_i)^2}{n}}{n-1}}$.

	x_i	x_i^2
	50	2500
	37	1369
	29	841
	54	2961
	30	900
	61	3721
	47	2209
	38	1444
	34	1156
	61	3721
Sum =	441	20777

In this problem we know that $n = 10$.

We compute the mean with the information from the table.

$$\bar{x} = \frac{\sum x_i}{n} = \frac{441}{10} = 44.1$$

We compute the sample standard deviation with the information from the table.

$$s = \sqrt{\frac{\sum x_i^2 - \frac{(\sum x_i)^2}{n}}{n-1}}$$

$$s = \sqrt{\frac{20777 - \frac{(441)^2}{10}}{10-1}}$$

$$s = \sqrt{\frac{20777 - \frac{194481}{10}}{9}}$$

$$s = \sqrt{\frac{20777 - 19448.1}{9}}$$

$$s = \sqrt{\frac{1328.9}{9}}$$

$$s = \sqrt{147.6556}$$

$$s = 12.15$$

For sample data that we get in a frequency distribution, to compute the standard deviation we

1. List the data classes in the first column. If required determine the midpoint of the classes and consider this the first column.
2. In the second column list the frequencies of the classes and determine the sum $\sum f_i$
3. In the third column compute the products of the frequencies time the data value (midpoint) and determine the sum of the third column. $\sum f_i x_i$
4. In the fourth column determine the squares of the data values (midpoint),
5. In the fifth column determine the products of the frequencies times the data value (midpoint) and determine their sum $\sum f_i x_i^2$
6. Substitute into the formula for sample or population and simplify.

EXAMPLE:

Twenty runners who finished a 10 kilometer race were asked the question “How many miles a week do you run? Their responses are summarized in the grouped frequency distribution below. Determine the mean and standard deviation for this sample.

Class	Frequency
5.5 – 10.5	1
10.5 – 15.5	2
15.5 – 20.5	3
20.5 – 25.5	5
25.5 – 30.5	4
30.5 – 35.5	3
35.5 – 40.5	2

Following the directions given above, we set-up a table of values.

Class	Midpoint x_i	Frequency f_i	$x_i f_i$	x_i^2	$x_i^2 f_i$
5.5 – 10.5	8	1	8	64	64
10.5 – 15.5	13	2	26	169	338
15.5 – 20.5	18	3	54	324	972
20.5 – 25.5	23	5	115	529	2645
25.5 – 30.5	28	4	112	784	3126
30.5 – 35.5	33	3	99	1089	3267
35.5 – 40.5	38	2	76	1444	2888
Sums =		20	490		13310

We can compute the mean using the formula

$$\bar{x} = \frac{\sum x_i \cdot f_i}{\sum f_i} = \frac{490}{20} = 24.5$$

The standard deviation is computed using the formula

$$s = \sqrt{\frac{\sum f_i x_i^2 - \frac{(\sum f_i x_i)^2}{\sum f_i}}{(\sum f_i) - 1}}$$

$$s = \sqrt{\frac{13310 - \frac{(490)^2}{20}}{(20) - 1}}$$

$$s = \sqrt{\frac{13310 - \frac{240100}{20}}{19}}$$

$$s = \sqrt{\frac{13310 - 12005}{19}}$$

$$s = \sqrt{\frac{1305}{19}}$$

$$s = \sqrt{68.68421053}$$

$$s = 8.28756$$

$$s = 8.3$$

Section 4.3.9 Inter-quartile Calculations

A quartile is a position in an arranged, from lowest to highest, data set that has exactly 25%, 50% or 75% of the data below that point. Every data set had three quartile points: a first quartile, a second quartile (called the median) and a third quartile. The quartiles points divide the data set into four parts each containing 25% of the data.

The procedure we use to compute the depth of the median, $d(m) = 1 + \frac{1}{2}(n-1)$, can be generalized to calculate the first and the third quartiles. We use the symbols Q_1 and Q_3 to represent the data values one quarter and three quarters of the way through the data respectively.

The depth of Q_1 (called the first quartile) is given by the formula $d(Q_1) = 1 + \frac{1}{4}(n-1)$, and the

depth of Q_3 (called the third quartile) is given by the formula $d(Q_3) = 1 + \frac{3}{4}(n-1)$.

One last statistic that we may have interest in computing is the inter-quartile range. The inter-quartile range tells us the range of the middle 50% of all the data. To compute the inter-quartile range we determine the difference of the third quartile minus the first quartile or $Q_3 - Q_1$.

EXAMPLE:

Compute the depth of the first and third quartile if $n = 16$.

To compute the depth of Q_1 and Q_3 we can use the formulas

$$d(Q_1) = 1 + \frac{1}{4}(n-1) \qquad d(Q_3) = 1 + \frac{3}{4}(n-1)$$

$$d(Q_1) = 1 + \frac{1}{4}(16-1) \qquad d(Q_3) = 1 + \frac{3}{4}(16-1)$$

$$d(Q_1) = 1 + \frac{1}{4}(15) \qquad d(Q_3) = 1 + \frac{3}{4}(15)$$

$$d(Q_1) = 1 + 3.75 \qquad d(Q_3) = 1 + 11.25$$

$$d(Q_1) = 4.75 \qquad d(Q_3) = 12.25$$

Now that we can compute the depth of a quartile, the next step is to compute the actual quartile values. To compute these values we will do just like we did with the median except that there are some slight differences.

To compute quartiles,

1. Arrange all of the data values in order from low to high.
2. Determine how many data values, n , are in the data set.
3. Use the formula to determine the position or depth of the quartile in the data

$$\text{set: } d(Q_1) = 1 + \frac{1}{4}(n-1) \text{ for } d(Q_1) \text{ and } d(Q_3) = 1 + \frac{3}{4}(n-1) \text{ for } d(Q_3)$$

- iii. If the depth is a whole number.
Begin at the low end of the data and count each data value until you get to the $d(Q_1)$ or $d(Q_3)$ position. This is the first quartile or third quartile respectively.

- iv. If $d(Q_1)$ or $d(Q_3)$ is a decimal then the quartile is located between two data values.
Beginning at the low end of the data, count each data value until you get to the whole number part of the decimal for the depth.
Multiply the difference of the two numbers either side of the depth position by the decimal part of the depth. This will be either 0.25 or 0.5 or 0.75
Add this to the number below the depth position. The result is the appropriate quartile.

EXAMPLE:

You work in an area of a plant that you consider to be extremely noisy and are aware of noise induced hearing loss. This type of loss can occur from 8 hours of exposure at a sound intensity of 85 decibels, a measure of sound pressure against the eardrum. A running gas lawn mower is about 85-80 decibel. Sound levels above 120 decibels, a jack-hammer, for a shorter period of time can also cause permanent loss. You complain to the plant's public safety officer who the monitors the sound in your area for 15 days. The peak decibel readings in your area as follow:

100 59 78 97 84 64 53 59 89 88 94 66 57 62 64

Compute the first quartile, median, and third quartile sound levels.

We arrange the data in order to get

53 57 59 59 62 64 64 66 78 84 88 89 94 97 100

There are 15 data values so $n = 15$.

We now compute $d(Q_1)$

$$d(Q_1) = 1 + \frac{1}{4}(n-1) = 1 + \frac{1}{4}(15-1) = 1 + \frac{1}{4}(14) = 1 + 3.5 = 4.5$$

Next we determine Q_1 . In the ordered data set 4.5 places is between 59 and 62. The difference of these two numbers is 3 and 0.5 times 3 is 1.5. We add 1.5 to the lower of the two numbers, 59, and get 60.5. The first quartile is 60.5.

We now compute the median

$$d(M) = 1 + \frac{1}{2}(n-1) = 1 + \frac{1}{2}(15-1) = 1 + \frac{1}{2}(14) = 1 + 7 = 8$$

Next we determine the median. In the ordered data set 8 places is 66. The median is 66.

We now compute $d(Q_3)$

$$d(Q_3) = 1 + \frac{3}{4}(n-1) = 1 + \frac{3}{4}(15-1) = 1 + \frac{3}{4}(14) = 1 + 10.5 = 11.5$$

Next we determine Q_3 . In the ordered data set 11.5 places is between 88 and 89. The difference of these two numbers is 1 and 0.5 times 1 is 0.5. We add 0.5 to the lower of the two numbers, 88, and get 88.5. The third quartile is 88.5.

The inter-quartile range can be easily determined once we know Q_3 and Q_1 . For this example we take $88.5 - 60.5 = 28.5$. The range of the middle 50% of all the data is 28.5

Section 4.3.10 Box Plots

So what does that information tell us about dangerous levels of noise? With a third quartile of 88.5 decibels we are safe 88.5 % of the time and are exposed to dangerous levels 25% of the time. We would like to display this information graphically. A graph that displays this information is called a “box plot” or a “box and whiskers plot.” This plot is based upon a 5 number summary. The 5 numbers in the summary are the low data value, the first quartile, the median, the third quartile, and the high data value,

The box plot in its simplest form is a scaled diagram with a straight line (whisker) drawn from the minimum score to Q_1 . From Q_1 to Q_3 a box is drawn with the position of the median marked. A second whisker then drawn from Q_3 to the maximum score. This construction generates a graphical representation of the distribution of the data set.

To construct a 5-number summary so that we can draw a box plot we must

1. Order the data from low to high
2. Determine the low data value
3. Determine the first quartile
4. Determine the median
5. Determine the third quartile
6. determine the high data value
7. Draw the plot.

EXAMPLE:

For the decibel problem above draw the box plot of that data. The data was as follow:

100 59 78 97 84 64 53 59 89 88 94 66 57 62 64

Arrange the data from low to high.

53 57 59 59 62 64 64 66 78 84 88 89 94 97 100

Determine the low data value: 53. We got this from looking at the arranged list.

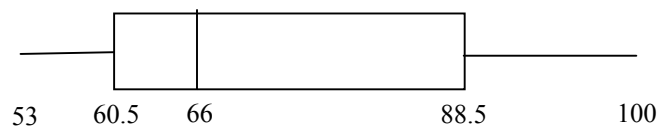
Determine the first quartile: 60.6. We computed this in the last example

Determine the median: 66. We computed this in the last example

Determine the third quartile: 88.5. We computed this in the last example

Determine the high data value: 100. We got this from looking at the arranged list.

Draw the plot:



Your Turn!!

1. A simple maze was constructed to determine the amount of time to the nearest second that it would take a rat to find the food at the end of the maze. Twelve rats were tested. The time it took each rat is given below.

18.1 , 19.7 , 17.8 , 19.1 , 16.5 , 20.0 , 18.5 , 17.9 , 20.9 , 20.3 , 19.4 , 18.3

Compute:

- The mean, \bar{x}
 - The median
 - The mode, if one exists
 - The first quartile, Q_1
 - The third quartile, Q_3
 - The inter-quartile range
 - The range
 - The sample standard deviation, s
 - The population standard deviation, σ
 - Draw a box and whiskers plot for this data
2. The following frequency distribution is based on the test scores on the Statistics chapter of Intro to College Math for three sections taught by the same teacher last year.

X	f	X	f	X	f
69	1	79	4	89	3
70	0	80	3	90	0
71	0	81	4	91	2
72	2	82	5	92	1
73	1	83	6	93	0
74	3	84	7	94	1
75	1	85	7	95	0
76	0	86	9	96	1
77	3	87	4	97	1
78	2	88	4		

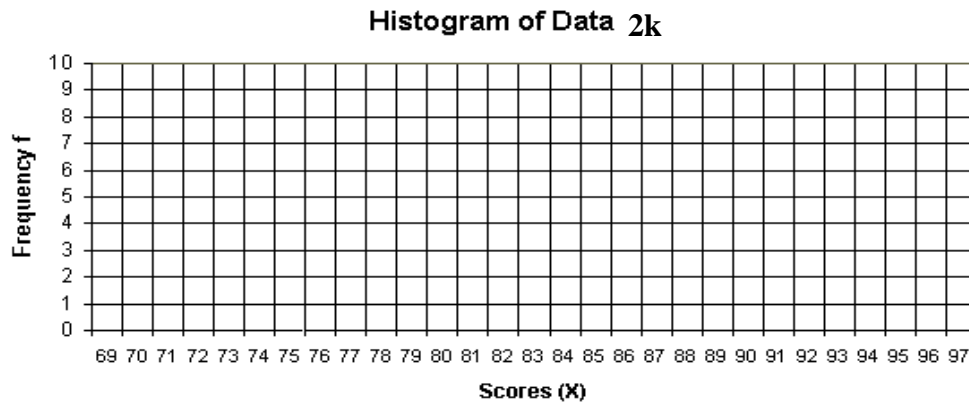
Compute:

- The mean, \bar{x}
- The median
- The mode, if one exists
- The first quartile, Q_1
- The third quartile, Q_3
- The inter-quartile range

- g. The range
- h. The sample standard deviation, s
- i. The population standard deviation, σ
- j. Draw a box and whiskers plot for this data



- k. Draw a histogram of this data set.



Chapter 4 Practice Exam

1. What is statistics?
2. What kind of data is data that is said to be quantitative?
3. During a census, one of the questions you are asked to respond to is “What type of housing do you have?” The choices are apartment (A), condominium (C), House (H), or Mobile Home (M). Fifty people from the town of Happyville responded as follows:

H	C	H	M	H	A	C	A	M	C
M	C	A	M	A	C	C	M	C	C
H	A	H	H	M	H	C	C	C	M
C	H	C	H	M	A	A	H	M	H
A	A	H	C	C	M	A	C	M	M

- a. Form a frequency distribution for this data
- b. Create a bar chart for the data.
- c. Create a circle graph for the data.

4. A cashier was bored one day while working at the Quick Stuff Mart. To pass the time the cashier recorded the number of purchases a customer made other than gas. The next 30 people who bought something besides gas at the store purchased the following number of items:

2	9	4	3	6	6	2	8	6	5
7	5	3	8	6	6	2	3	2	4
6	9	9	8	9	4	2	1	7	4

Compute:

- a. The mean, \bar{x}
 - b. The median
 - c. The mode, if one exists
 - d. The first quartile, Q_1
 - e. The third quartile, Q_3
 - f. The inter-quartile range
 - g. The range
 - h. The sample standard deviation, s
 - i. The population standard deviation, σ
 - j. Draw a bar chart for this data
 - k. What is the relative frequency for the 7 purchase class
 - l. Draw a cumulative graph for this data
 - m. Draw a box and whiskers plot for this data
5. The Pittsburg Tribune Review reported the murder rates 25 selected but unnamed cities in the United States. The murder rates they reported were as follows:

248	270	366	149	109	348	71	73	68
598	74	226	241	73	278	514	41	46
63	69	597	39	34	65	27		

Compute:

- a. The mean, \bar{x}
- b. The median
- c. The mode, if one exists
- d. The first quartile, Q_1
- e. The third quartile, Q_3
- f. The inter-quartile range
- g. The range
- h. The sample standard deviation, s
- i. The population standard deviation, σ

- j. Draw a box and whiskers plot for this data
 - k. Create a grouped frequency distribution with 7 classes. Draw a histogram of the data
6. The National Association of Theater Owners has kept track of the number of indoor movie theaters in the United States since 1987. The year and number of theaters is given below.

Year	Number	Year	Number	Year	Number	Year	Number
1987	20595	1993	24798	1999	36448	2005	37092
1988	21632	1994	25830	2000	35567	2006	37776
1989	21907	1995	26995	2001	34490	2007	38159
1990	22904	1996	28905	2002	35170		
1991	23740	1997	31050	2003	35361		
1992	24344	1998	33418	2004	36012		

- a. Create a time series graph of the data
 - b. Does the graph reveal any trends in the data?
 - c. Have there been any years in which the number of theaters declined? What years?
7. A truck rental company charges customers a flat fee of \$95 a day for use of a truck and \$0.45 per mile driven after the first 100 miles.
- a. What is the cost if you rent the truck for 1 day and drive 75 miles?
 - b. What is the cost if you rent the truck for 3 days and drive 75 miles?
 - c. What is the cost if you rent the truck for 1 day and drive 220 miles?
 - d. What is the cost if you rent the truck for 2 day and drive 160 miles?
 - e. Let c be the cost of the rental and m be the number of miles driven, then the cost equation for a one day rental is $c = 95$, if the miles driven are less than 100 and $c = 95 + 0.45(m - 100)$ if the miles driven is over 100 miles.
 - i. Draw the graph of the cost equation.
 - ii. If you drive 210 miles what is the cost?
 - iii. If the cost was \$320 how far did you drive?

Chapter 4 Answer Section

Section 4.1 Solution Set:

1. Covance Ages

a. Ungrouped frequency distribution

Age	Tally	Frequency
18		6
19		9
20		4
21		8
22		8
23		5
24		6
25		10
26		4
27		8
28		7
29		7
30		4
31		5
32		7
33		4
34		1
35		3
36		0
37		0
38		1

b. Grouped frequency distribution

Age	Tally	Frequency
18 – 20		19
21 – 23		21
24 – 27		28
28 – 30		18
31 – 32		12
33 – 35		8
36 – 38		1

- c. The answer is subject to interpretation. There are many answers possible in the context of what type of information is being sought after from the data.

2. Garbage weights

- a. Grouped frequency distribution.

Weight	Tally	Frequency
4.1 – 11.0		4
11.1 – 18.0		10
18.1 – 25.0		17
25.1 – 32.0		11
32.1 – 39.0		9
39.1 – 46.0		6
46.1 – 53.0		4
53.1 – 60.0		1

- b. Cumulative frequency distribution

Weight	Tally	Frequency	Cumulative Frequency
4.1 – 11.0		4	4
11.1 – 18.0		10	14
18.1 – 25.0		17	31
25.1 – 32.0		11	42
32.1 – 39.0		9	51
39.1 – 46.0		6	57
46.1 – 53.0		4	61
53.1 – 60.0		1	62

3. M & M's

- a. Frequency Distribution

Color	Tally	Frequency
Blue		15
Brown		34
Green		8
Orange		9
Red		23
Yellow		26

b. Relative frequency distribution

Color	Tally	Frequency	Relative Frequency
Blue		15	0.13
Brown		34	0.30
Green		8	0.07
Orange		9	0.08
Red		23	0.20
Yellow		26	0.23

4. a. Grouped frequency distribution

Age	Tally	Frequency
21 – 30		28
31 – 40		30
41 – 50		12
51 – 60		2
61 – 70		2
71 – 80		2

b. Cumulative frequency distribution

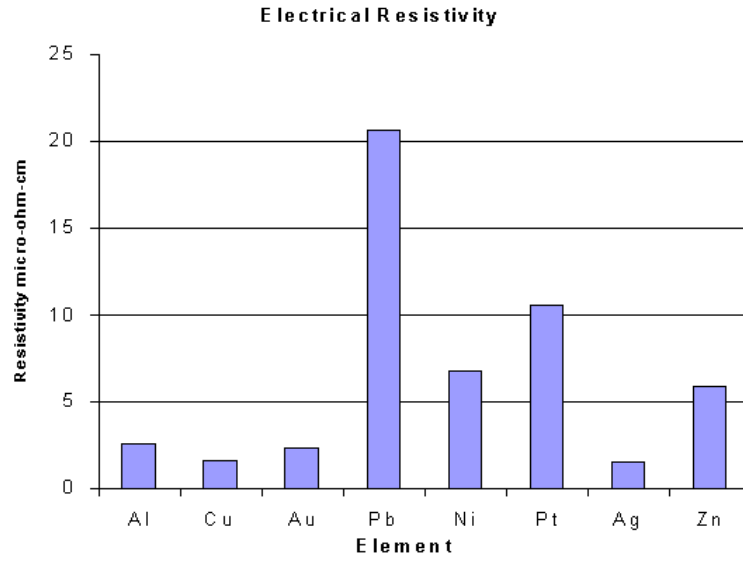
Weight	Tally	Frequency	Cumulative Frequency
21 – 30		28	28
31 – 40		30	58
41 – 50		12	70
51 – 60		2	72
61 – 70		2	74
71 – 80		2	76

d. Relative frequency distribution

Weight	Tally	Frequency	Relative Frequency
21 – 30		28	0.37
31 – 40		30	0.39
41 – 50		12	0.16
51 – 60		2	0.03
61 – 70		2	0.03
71 – 80		2	0.03

Section 4.2 Solution Set:

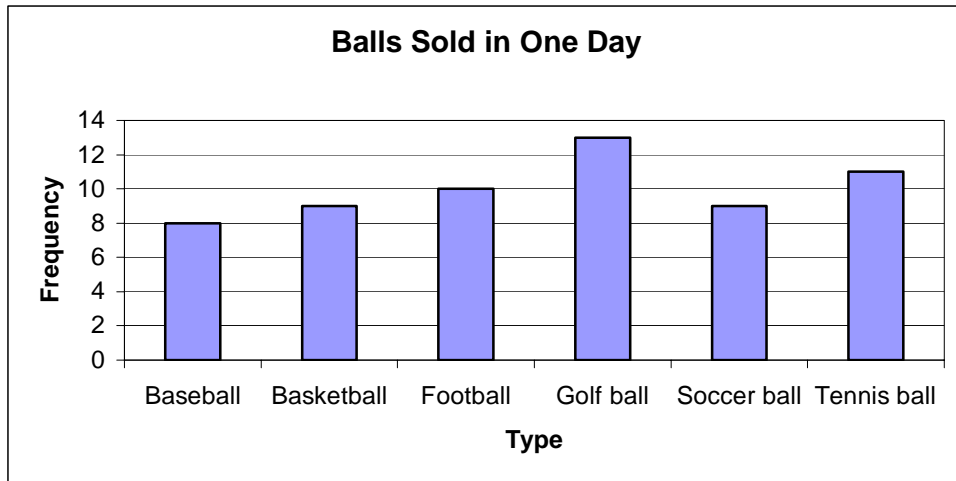
1a.



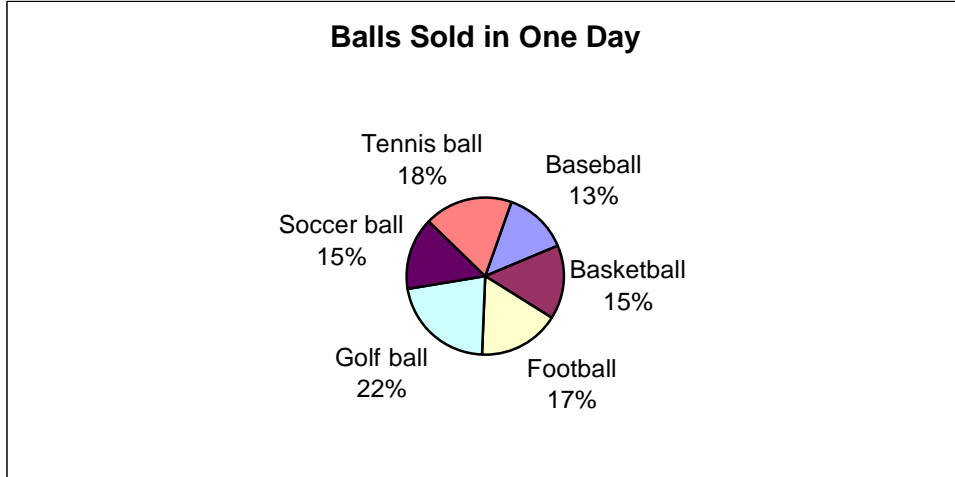
1b. Silver (Ag)

1c. Lead (Pb)

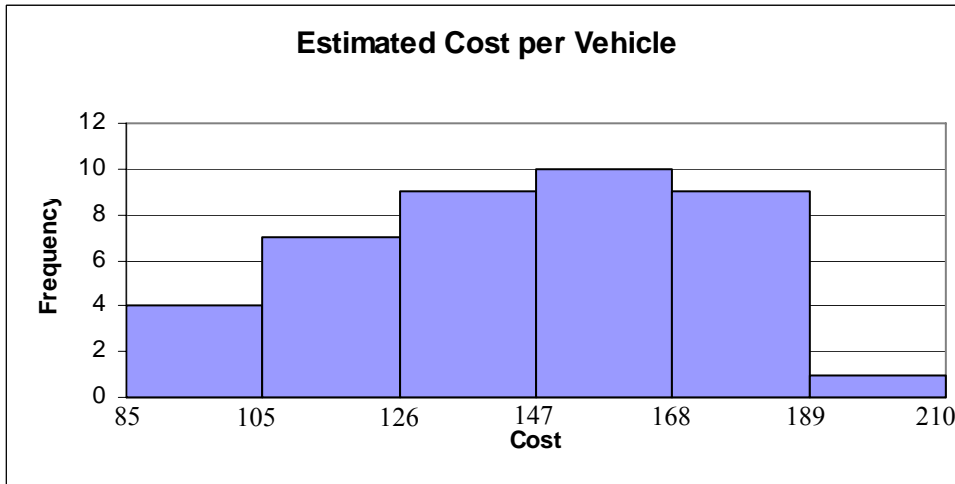
2a.



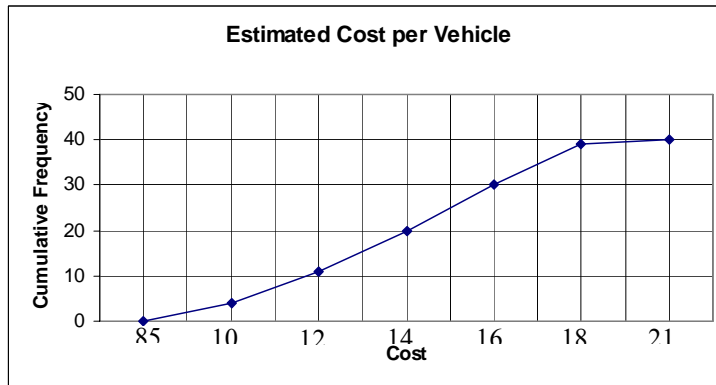
2b.



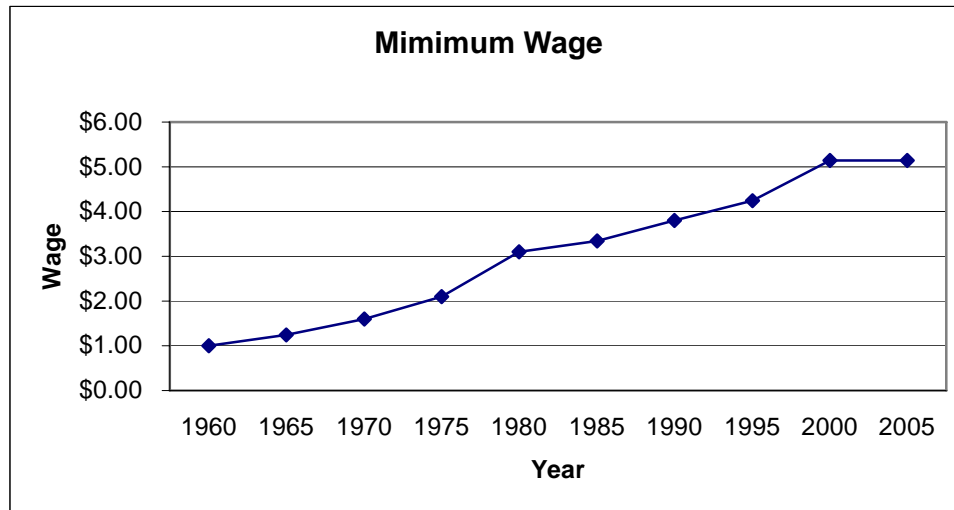
3a.



3b.



4a.



4b. 2000 or 2005

4c. 1975 – 1980

5a. ~ 29 m/s

5b. ~ 5.1 sec

5c. ~9.8 m/s²

5d. ~98 m/s

6a. $y = -1$ 6b. $y = 6$ 6c. $y = 15$ 6d. $y = -10$ 6e. $y = -5$ 6f. $-2 < x < 0$ and $x > 2$ 6g. $x < -2$ and $0 < x < 2$ 6h. $x = -2$ and $x = 2$ 6i. $x = 0$

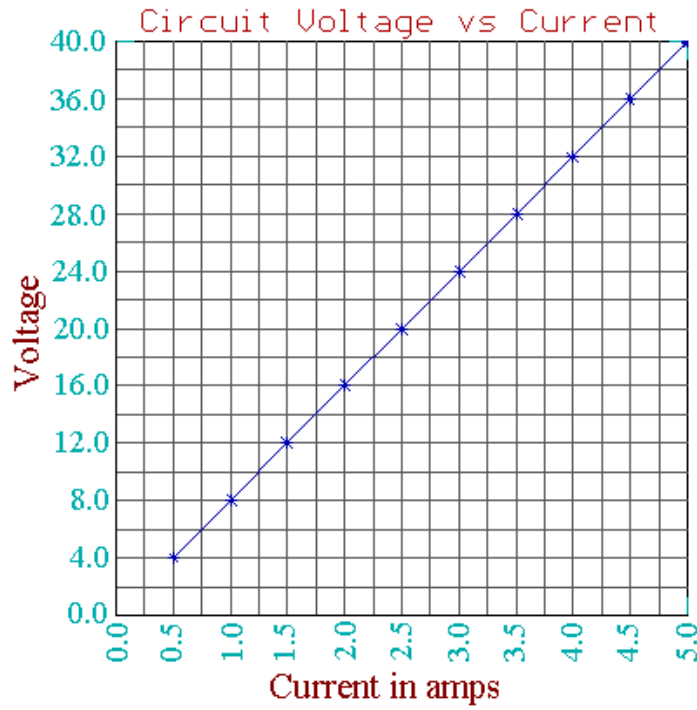
7a. 0 mph

7b. 0 mph

7c. ~ 45 mph

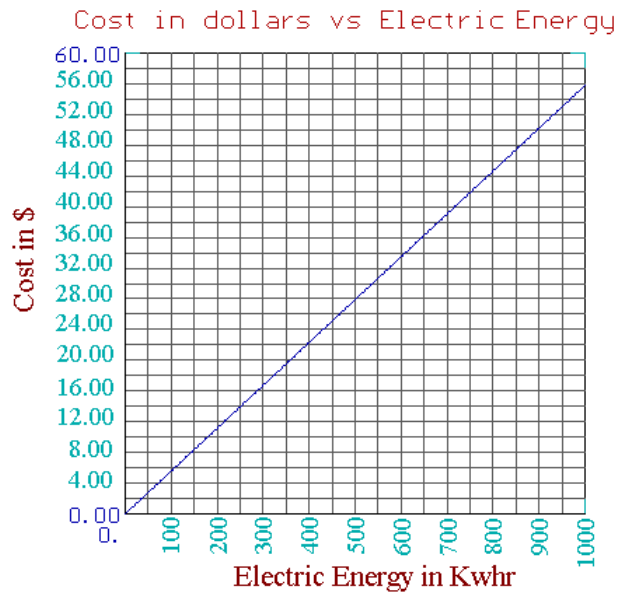
7d. 40 sec

8a.



8b. 40.0 volts

9a.



9b. \$33.48

Section 4.3 Solution Set:

Computational table of data

i	x_i	x_i^2
1	16.5	272.25
2	17.8	316.84
3	17.9	320.41
4	18.1	327.61
5	18.3	334.89
6	18.5	342.25
7	19.1	364.81
8	19.4	376.36
9	19.7	388.09
10	20.0	400
11	20.3	412.09
12	20.9	436.81
Sum	226.5	4292.41

1a. Mean:

$$\bar{x} = \frac{\sum x_i}{n} = \frac{226.5}{12} = 18.875$$

1b. Median

$$d(M) = 1 + \frac{1}{2}(n-1) = 1 + 5.5 = 6.5$$

$$M = 18.5 + \frac{1}{2}(19.1 - 18.5) = 18.5 + \frac{1}{2}(0.6) = 18.5 + 0.3 = 18.8$$

1c. There is no mode. No score appears more than any other score.

1d. First Quartile

$$d(Q_1) = 1 + \frac{1}{4}(n-1) = 1 + 2.75 = 3.75$$

$$Q_1 = 17.9 + .75(18.1 - 17.9) = 18.05$$

1e. Third Quartile

$$d(Q_3) = 1 + .75(n-1) = 1 + 8.25 = 9.25$$

$$Q_3 = 19.7 + 0.25(20.0 - 19.7) = 19.775$$

1f. Inter-quartile range

$$IQR = Q_3 - Q_1 = 19.775 - 18.05 = 1.725$$

1g. Range

$$R = H - L = 20.9 - 16.5 = 4.4$$

1h. Sample standard deviation

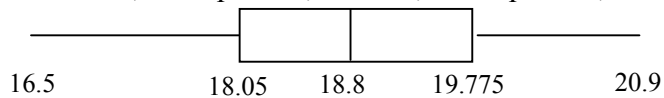
$$s = \sqrt{\frac{\sum x_i^2 - \frac{(\sum x_i)^2}{n}}{n-1}} = \sqrt{\frac{4292.41 - \frac{226.5^2}{12}}{11}} = 1.2513$$

1i. Population standard deviation

$$\sigma = \sqrt{\frac{\sum x_i^2 - \frac{(\sum x_i)^2}{n}}{n}} = \sqrt{\frac{4292.41 - \frac{226.5^2}{12}}{12}} = 1.1980$$

1j. Box and whiskers plot

The low, first quartile, median, third quartile, and high were computed above.



2. Computational table of data

x (Score)	f (Frequency)	<i>Cumulative</i> <i>frequency</i>	$f * x$	x^2	$f * x^2$
69	1	1	69	4761	4761
70	0	1	0	4900	0
71	0	1	0	5041	0
72	2	3	144	5184	10368
73	1	4	73	5329	5329
74	3	7	222	5476	16428
75	1	8	75	5625	5625
76	0	8	0	5776	0
77	3	11	231	5929	17787
78	2	13	156	6084	12168
79	4	17	316	6241	24964
80	3	20	240	6400	19200
81	4	24	324	6561	26244
82	5	29	410	6724	33620
83	6	35	498	6889	41334
84	7	42	588	7056	49392
85	7	49	595	7225	50575
86	9	58	774	7369	66564
87	4	62	348	7569	30276
88	4	66	352	7744	30976
89	3	69	267	7921	23763
90	0	69	0	8100	0

91	2	71	182	8281	16562
92	1	72	92	8464	8464
93	0	73	0	8649	0
94	1	73	94	8836	8836
95	0	74	0	9025	0
96	1	75	96	9216	9216
97	1		97	9409	9409
Sum	75		6243		521861

2a. Mean:

$$\bar{x} = \frac{\sum f_i \cdot x_i}{\sum f_i} = \frac{6243}{75} = 83.24$$

2b. Median

$$d(M) = 1 + \frac{1}{2}(n-1) = 1 + 37 = 38$$

The number 38 is less than 42 so the median is 84

2c. Mode: Highest frequency is 9. The mode score is 86.

2d. First Quartile

$$d(Q_1) = 1 + \frac{1}{4}(n-1) = 1 + 18.5 = 19.5$$

$$Q_1 = 80 + .50(80 - 80) = 80$$

2e. Third Quartile

$$d(Q_3) = 1 + .75(n-1) = 1 + 55.5 = 56.5$$

$$Q_3 = 86 + 0.50(86 - 86) = 86$$

2f. Inter-quartile range

$$IQR = Q_3 - Q_1 = 86 - 80 = 6$$

2g. Range

$$R = H - L = 97 - 69 = 28$$

2h. Sample standard deviation

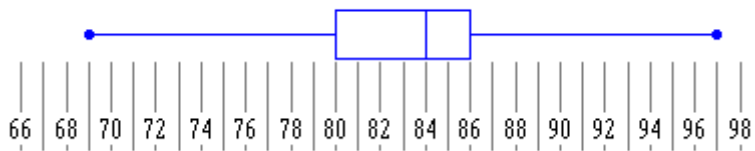
$$s = \sqrt{\frac{\sum f_i \cdot x_i^2 - \frac{(\sum f_i \cdot x_i)^2}{\sum f_i}}{(\sum f_i) - 1}} = \sqrt{\frac{521861 - \frac{(6243)^2}{75}}{75 - 1}} = 5.4447$$

2i. Population standard deviation

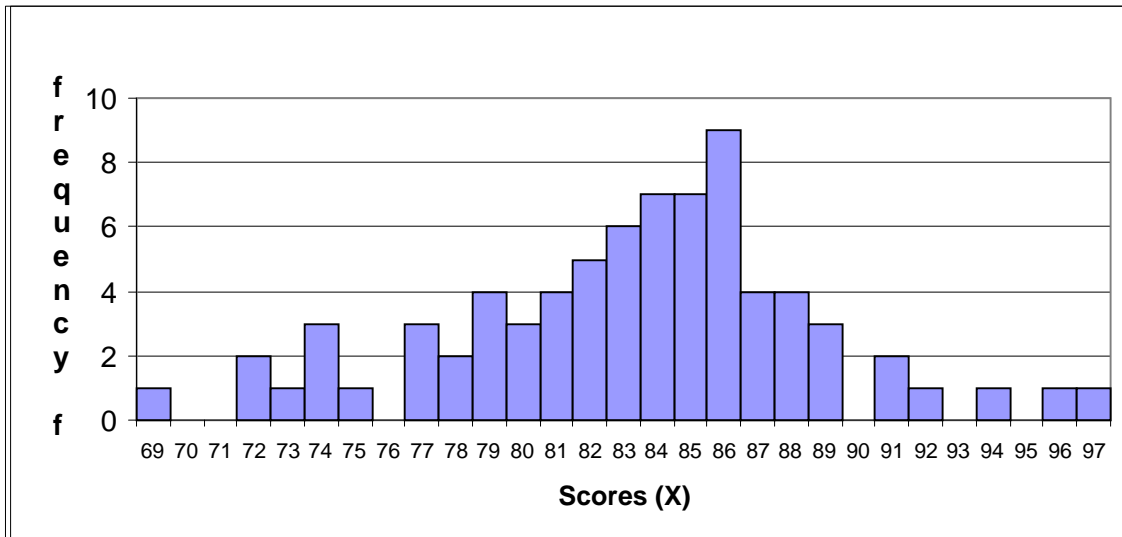
$$\sigma = \sqrt{\frac{\sum f_i \cdot x_i^2 - \frac{(\sum f_i \cdot x_i)^2}{\sum f_i}}{\sum f_i}} = \sqrt{\frac{521861 - \frac{(6243)^2}{75}}{75}} = 5.4082$$

2j. Box and whiskers plot

The low, first quartile, median, third quartile, and high were computed above.



2k.



Chapter 4 Practice Exam Solution Set:

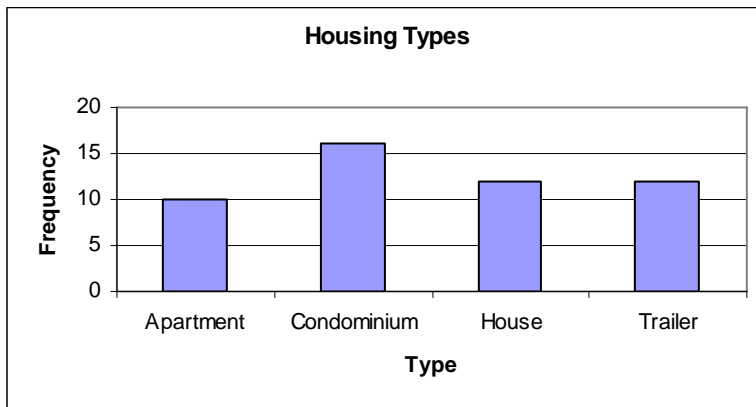
1. Statistics is the science of collecting, organizing, analyzing, presenting, and interpreting data.

2. Data that is quantitative is data that contains numbers upon which arithmetical operations can be performed.

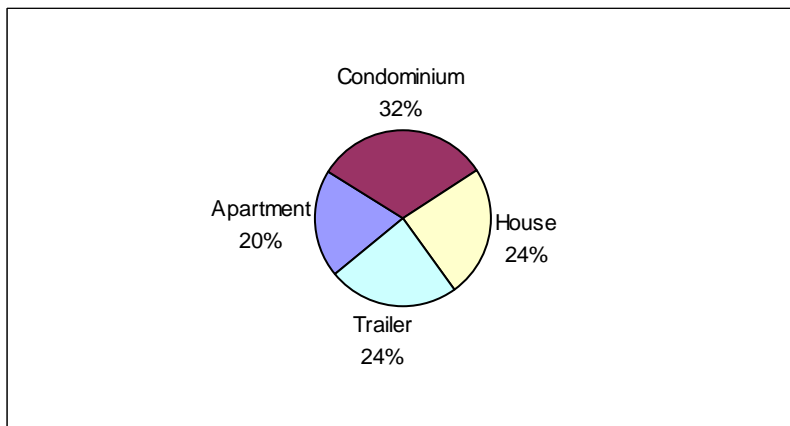
3a. Frequency Distribution

Type	Tally	Frequency
Apartment		10
Condominium		16
House		12
Trailer		12

3b. Bar Chart



3c. Circle chart



4. Computational table for data.

x	f (Frequency)	<i>Cumulative</i> <i>frequency</i>	$f * x$	x^2	$f * x^2$
1	1	1	1	1	1
2	5	6	10	4	20
3	3	9	9	9	27
4	4	13	16	16	64
5	2	15	10	25	50
6	6	21	36	36	216
7	2	23	14	49	98
8	3	26	24	64	192
9	4	30	36	81	324
Sum	30		156		992

4a. Mean:

$$\bar{x} = \frac{\sum f_i \cdot x_i}{\sum f_i} = \frac{156}{30} = 5.2$$

4b. Median

$$d(M) = 1 + \frac{1}{2}(n-1) = 1 + \frac{1}{2}(30-1) = 1 + 14.5 = 15.5$$

The 15th data element is a 5 and the 16th data element is a 6. We average these two numbers, $\frac{5+6}{2} = \frac{11}{2} = 5.5$

4c. Mode: Highest frequency is 6. The mode score is 6

4d. First Quartile

$$d(Q_1) = 1 + \frac{1}{4}(n-1) = 1 + \frac{1}{4}(30-1) = 1 + \frac{1}{4}(29) = 1 + 7.25 = 8.25$$

$$Q_1 = 3 + .25(3-3) = 3$$

4e. Third Quartile

$$d(Q_3) = 1 + \frac{3}{4}(n-1) = 1 + \frac{3}{4}(30-1) = 1 + \frac{3}{4}(29) = 1 + 21.75 = 22.75$$

$$Q_3 = 7 + 0.75(7-7) = 7$$

4f. Inter-quartile range

$$IQR = Q_3 - Q_1 = 7 - 3 = 4$$

4g. Range

$$R = H - L = 9 - 1 = 8$$

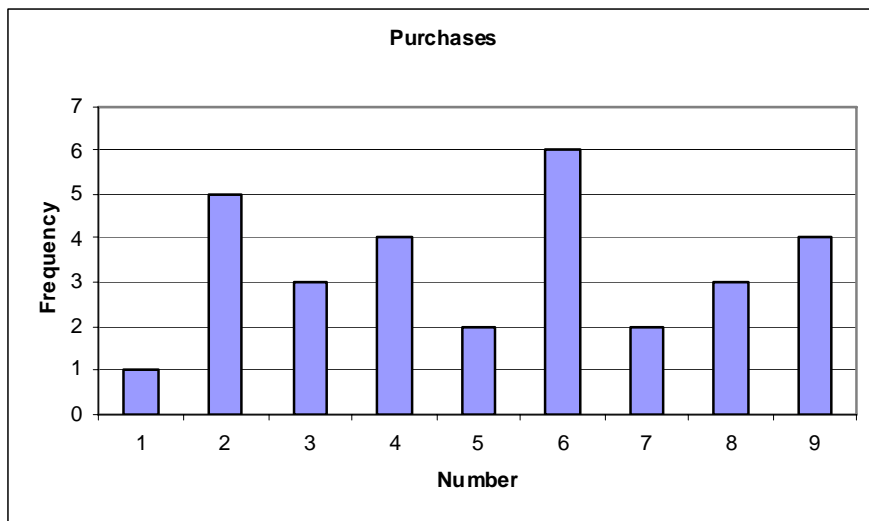
4h. Sample standard deviation

$$s = \sqrt{\frac{\sum f_i \cdot x_i^2 - \frac{(\sum f_i \cdot x_i)^2}{\sum f_i}}{(\sum f_i) - 1}} = \sqrt{\frac{992 - \frac{(156)^2}{30}}{30 - 1}} = \sqrt{\frac{992 - \frac{(156)^2}{30}}{30 - 1}} = 2.4969$$

4i. Population standard deviation

$$\sigma = \sqrt{\frac{\sum f_i \cdot x_i^2 - \frac{(\sum f_i \cdot x_i)^2}{\sum f_i}}{\sum f_i}} = \sqrt{\frac{992 - \frac{(156)^2}{30}}{30}} = \sqrt{\frac{992 - \frac{24336}{30}}{30}} = 2.4549$$

4j.



4k. The relative frequency for 7 purchases is 0.07

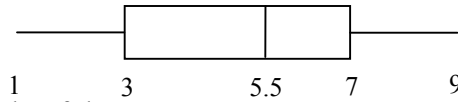
<i>x</i>	<i>f</i> (Frequency)	<i>Relative frequency</i>
1	1	0.03
2	5	0.17
3	3	0.10
4	4	0.13
5	2	0.07
6	6	0.20
7	2	0.07
8	3	0.10
9	4	0.13
Sum	30	1.00

4l.



4m. Box and whiskers plot

The low, first quartile, median, third quartile, and high were computed above.



5. Computational table of data

x	x^2		x	x^2		x	x^2		x	x^2
27	729		68	4624		149	22201		366	133956
34	1156		69	4761		226	51076		514	264196
39	1521		71	5041		241	58081		597	356409
41	1681		73	5329		248	61504		598	357604
46	2116		73	5329		270	72900			
63	3969		74	5476		278	77284		SUM	SUM
65	4225		109	11881		348	121104		4687	1634153

5a. Mean:

$$\bar{x} = \frac{\sum x_i}{n} = \frac{4687}{25} = 187.48$$

5b. Median

$$d(M) = 1 + \frac{1}{2}(n - 1) = 1 + 12 = 13$$

From the sorted data the 13th data element is 74

5c. The most commonly occurring data value is 73. It appears twice.

5d. First Quartile

$$d(Q_1) = 1 + \frac{1}{4}(n-1) = 1 + \frac{1}{4}(25-1) = 1 + 6 = 7$$

From the sorted data the 7th data element is 65

5e. Third Quartile

$$d(Q_3) = 1 + .75(n-1) = 1 + .75(25-1) = 1 + 18 = 19$$

From the sorted data the 19th data element is 270

5f. Inter-quartile range

$$IQR = Q_3 - Q_1 = 270 - 65 = 205$$

5g. Range

$$R = H - L = 598 - 27 = 571$$

5h. Sample standard deviation

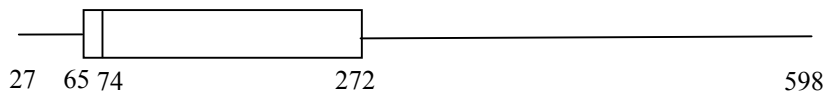
$$s = \sqrt{\frac{\sum x_i^2 - \frac{(\sum x_i)^2}{n}}{n-1}} = \sqrt{\frac{1634153 - \frac{4687^2}{25}}{24}} = 177.4160$$

5i. Population standard deviation

$$\sigma = \sqrt{\frac{\sum x_i^2 - \frac{(\sum x_i)^2}{n}}{n}} = \sqrt{\frac{1634153 - \frac{4687^2}{25}}{25}} = 173.8314$$

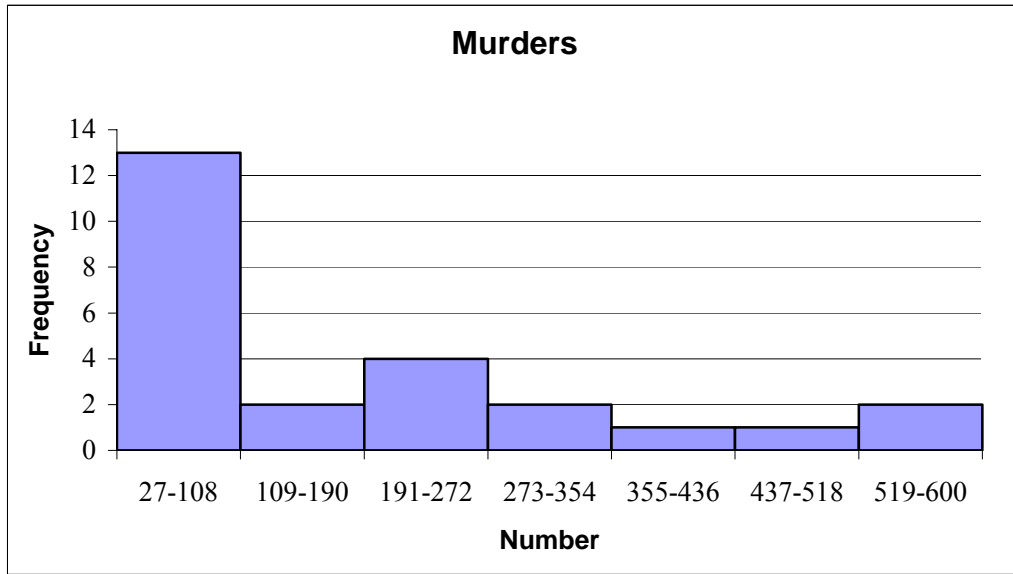
5j. Box and whiskers plot

The low, first quartile, median, third quartile, and high were computed above.



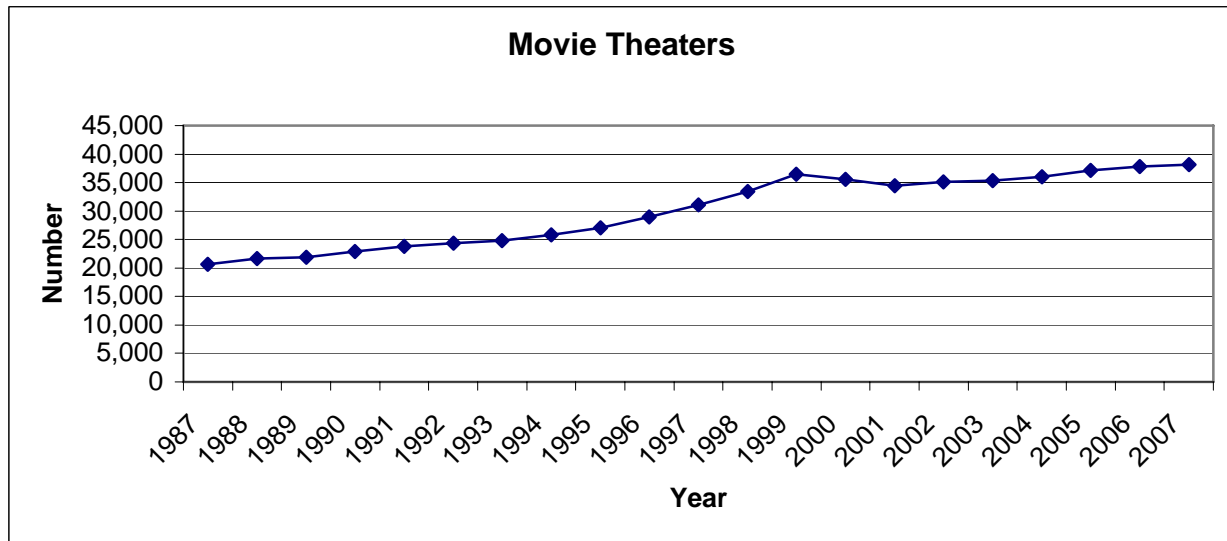
5k. Range: $R = H - L = 598 - 27 = 571$. Width: $\text{Range} \div 7 = 81.57$. We round to 82.

Low	High	Tally	Frequency
27	108		13
109	190		2
191	272		4
273	354		2
355	436		1
437	518		1
519	600		2



6. Time Series

6a.

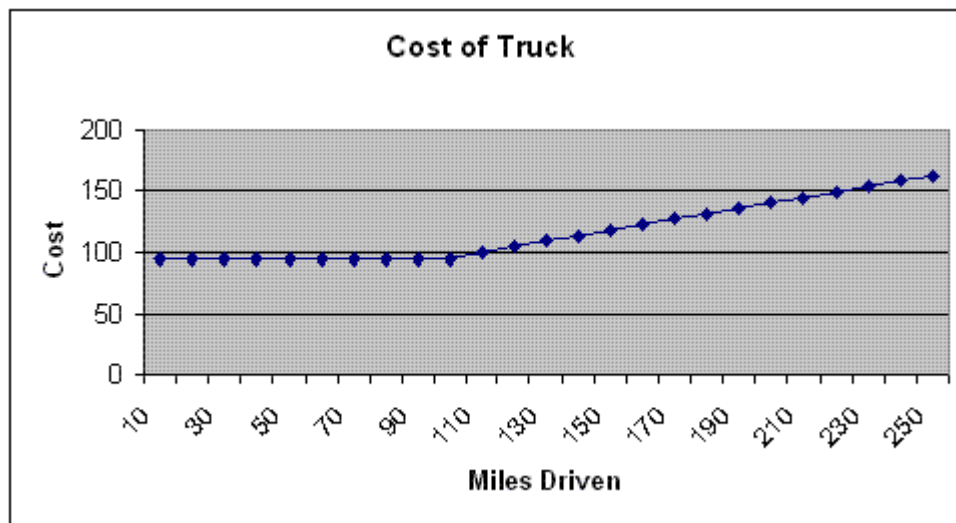


6b. The number of theaters is increasing over time.

6c. Yes. 1999 to 2000 and 2000 to 2001

7a. \$95. Because 75 is less than 100 there is no additional mileage charge. You pay only the flat fee.

- 7b. \$285. You pay 3 day of flat fee of \$95 and there is no extra mileage charge because 75 is less than 100.
- 7c. \$149 You pay the one day flat fee of \$95 plus the mileage charge of \$0.45 times 120 miles, the amount over 100 miles.
- 7d. \$217. You pay 2 days of flat fee of \$95 plus the mileage charge of \$0.45 times 60 miles, the amount over 100 miles.
- 7e i.



7e ii. \$144.50

$$c = 95 + 0.45(m - 100)$$

$$c = 95 + 0.45(210 - 100)$$

$$c = 95 + 0.45(110)$$

$$c = 95 + 49.5$$

$$c = 144.50$$

7e iii. 700 miles. You were charged for an additional 600 miles plus you had 100 miles as part of the base fee.

$$320 = 95 + 0.45(m - 100)$$

$$320 - 95 = 95 - 95 + 0.45(m - 100)$$

$$225 = 0.45(m - 100)$$

$$225 = 0.45m - 45$$

$$225 + 45 = 0.45m - 45 + 45$$

$$270 = 0.45m$$

$$\frac{270}{0.45} = \frac{0.45m}{0.45}$$

$$600 = m$$

Chapter 5: Measurement

In this chapter we will look at how to measure things using both the English System and the Metric System. When working with measurements it is important to understand that all measurements consist of two parts, a number part and a unit. For example, if we measure a table with a tape measure and report its width as 37.5 inches, 37.5 is the number part, inches are the unit part. The connection between the number part and unit part is the operation of multiplication. So when we write 37.5 inches, we really mean $(37.5)(1 \text{ in})$, i.e., 37.5 multiples of the base unit of one inch. The use of the word “base” in base unit does not imply that it is being raised to a power.

There are many different kinds of things we can measure, but all common physical measurements can be reduced to just a few kinds of things. The four basic quantities are often listed as follows:

1. **Length** which is measured in units such as feet, miles, meters, etc.
2. **Time** which is measured in units such as seconds, minutes, hours, etc.
3. **Mass** (or weight) which is measured in units such as grams, kilograms, pounds, etc.
4. **Electric charge** which is measured in coulombs or amp-seconds.

All other measurements can be expressed as a combination of these. It should really be noted that mass and weight are not identical. Weight is the gravitational force acting on a mass and varies with position in space. For example, the weight of an object on the surface of the earth is about six times the weight of that same object on the surface of the moon. However, the mass of the object is the same in both locations. Despite this difference, in these notes we will not distinguish between mass and weight.

Section 5.1: Linear Measurements

Measurement conversion is a necessary skill since the same set of units is often not used throughout a calculation. The basis of measurement conversion is the unit fraction.

A unit fraction is a fraction that has a value of 1. The catch is that 1 has infinitely many “aliases”. For example,

$$1 = \frac{12 \text{ in}}{1 \text{ ft}} = \frac{1 \text{ ft}}{12 \text{ in}} = \frac{4 \text{ qt}}{1 \text{ gal}} = \frac{1 \text{ gal}}{4 \text{ qt}} = \frac{1 \text{ min}}{60 \text{ sec}} = \frac{60 \text{ sec}}{1 \text{ min}} = \frac{1000 \text{ m}}{1 \text{ km}} = \frac{1 \text{ km}}{1000 \text{ m}}$$

All of these represent unit fractions, since the numerator is the same amount as the denominator.

When multiplying a quantity by a unit fraction (which has a value of one), the value of the quantity remains unchanged. That is for any quantity Q :

$$Q = Q \cdot 1$$

The trick is to use the “proper” aliases to cancel the units you don’t want and get the units you do want. For example, to convert 18 in to ft we use the unit fraction containing the ratio of the number of inches in a foot as follows:

$$18 \text{ in} = \frac{18 \cancel{\text{ in}}}{1} \times \frac{1 \text{ ft}}{12 \cancel{\text{ in}}} = \frac{18}{12} \text{ ft} = \frac{3}{2} \text{ ft} = 1\frac{1}{2} \text{ ft or } 1.5 \text{ ft}$$

As you can see, the number of inches is placed in the denominator of the unit fraction so that it appears in both the numerator and denominator and as a result cancels out.

Section 5.1.1: Conversions within the English System

As mentioned at the beginning of the chapter we will look at conversions within the English System and the Metric System. In this section, as in the example above, we will continue to look at conversions within the English System.

Example: *Convert 5 gallons to quarts.*

Solution: First, we must identify the necessary conversion factor between gallons and quarts. Since there are 4 quarts in 1 gallon, we set up a unit fraction using these values. We place the 1 gallon in the denominator of our unit fraction as follows because we are trying to “get rid of” the gallon units.

$$5 \text{ gallons} = \frac{5 \cancel{\text{ gallons}}}{1} \times \frac{4 \text{ quarts}}{1 \cancel{\text{ gallon}}} = 20 \text{ quarts}$$

Thus 5 gallons is equivalent to 20 quarts.

As an aide in setting up conversion calculations, a set of equivalent measurements is presented in the appendix at the end of this chapter.

There are times when we wish to convert measurements that are in decimal form to fractional form. The same technique using unit fractions works.

Example: *Round 0.567 in to the nearest 32nd of an inch.*

Solution: Since we are trying to convert to 32nd of an inch, the key is to use the unit fraction $\frac{32}{32}$.

$$0.567 \text{ in} = \frac{0.567 \text{ in}}{1} \times \frac{32}{32} = \frac{0.567 \times 32}{1 \times 32} = \frac{18.144}{32} \text{ in} \approx \frac{18}{32} \text{ in}$$

0.567 inches is eighteen thirty-seconds to the nearest thirty-seconds of an inch.

Example: Round 0.434 in to the nearest 64th of an inch.

Solution: Since we are trying to convert to 64th of an inch, the key is to use the unit fraction $\frac{64}{64}$.

$$0.434 \text{ in} = \frac{0.434 \text{ in}}{1} \times \frac{64}{64} = \frac{0.434 \times 64}{64} \text{ in} = \frac{27.776}{64} \text{ in} \approx \frac{28}{64} \text{ in} = \frac{7}{16} \text{ in}$$

So 0.434 in is seven sixteenth's of an inch to the nearest 64th of an inch.

For some problems we do not have a conversion factor that directly gives us the number of one type of unit in terms of the other. In these cases we may need to multiply by more than one unit fraction to get us from one unit to the other.

Example: Convert 190 ounces to gallons.

$$\begin{aligned} \text{Solution: } 190 \text{ oz} &= \frac{190 \cancel{\text{oz}}}{1} \times \frac{1 \cancel{\text{pt}}}{16 \cancel{\text{oz}}} \times \frac{1 \cancel{\text{qt}}}{2 \cancel{\text{pt}}} \times \frac{1 \text{ gal}}{4 \cancel{\text{qt}}} = \frac{190 \times 1 \times 1 \times 1}{1 \times 16 \times 2 \times 4} \text{ gal} \\ &= \frac{190}{128} \text{ gal} = \frac{95}{64} \text{ gal} = 1 \frac{31}{64} \text{ gal} = 1.484375 \text{ gal} \end{aligned}$$

190 ounces is $1 \frac{31}{64}$ gallons or 1.484375 gallons which is approximately 1.5 gallons.

In other more complicated conversions we may need to use more than one unit fraction because we have a measurement involving some type of rate where we need to convert both units involved in the rate.

Example: Convert 100 feet per second to miles per hour correct to one decimal place.

Solution: In this problem, we not only multiply by the unit fraction to convert the feet units to miles, but we must also multiply by the unit fractions that convert seconds to hours. As we do this we must make sure to set up the unit fractions so that the units we wish to “get rid of” are diagonal from each other so that they cancel.

$$\begin{aligned}
 100 \text{ ft/sec} &= \frac{100 \cancel{\text{ft}}}{1 \cancel{\text{sec}}} \times \frac{1 \text{ mile}}{5280 \cancel{\text{ft}}} \times \frac{60 \cancel{\text{sec}}}{1 \cancel{\text{min}}} \times \frac{60 \cancel{\text{min}}}{1 \text{ hr}} = \frac{100 \times 1 \times 60 \times 60}{1 \times 5280 \times 1 \times 1} \frac{\text{mile}}{\text{hr}} \\
 &= \frac{360,000 \text{ mi}}{5280 \text{ hr}} = \frac{2250 \text{ mi}}{33 \text{ hr}} = 68 \frac{6}{33} \text{ mi/hr} \approx 68.2 \text{ mi/hr}
 \end{aligned}$$

Section 5.1.2: The Metric System

One of the consequences of the French Revolution of 1789 was the development of the metric system of measurement. This system was designed to replace the earlier French system, which like its English counterpart had its origins in medieval society and royal institutions. Three features make the metric system very attractive. First, it is built on powers of 10, just like our decimal number system. Every unit is a multiple of 10 of some other unit. Thus, “strange” English multipliers like 3, 12, and 16 are banished! Second, a deliberate effort was made to coordinate different measures. For example, the fundamental unit of volume, the liter symbolized by L, is simply related to the fundamental unit of length, the meter symbolized by m, through the equation $1 \text{ m}^3 = 1000 \text{ L}$. Contrast this with the English system where $1 \text{ gal} = 231 \text{ in}^3 = 0.134 \text{ ft}^3$. The third advantage of the metric system is that it is “universal”. It can be used with **any** kind of measurement in the same way.

It is interesting to note that the metric system was so well accepted and in place when electrical measurements began some 150 years ago, only metric units were developed. The customary electric units we are all know, the volt (V), amp (A), and ohm (Ω) are all metric.

The metric system uses a two-part representation for all measurements. Each unit of measurement contains the base unit which is determined by what we are trying to measure. The important metric base units are as follows:

Base Unit	Symbol	What it Measures
meter	<i>m</i>	length
seconds	<i>s</i>	time
gram	<i>g</i>	mass
liter	<i>L</i>	volume
watt	<i>w</i>	power
joule	<i>j</i>	energy
newton	<i>N</i>	force
hertz	<i>Hz</i>	frequency
ampere	<i>A</i>	electric current
volt	<i>V</i>	electric potential
ohm	Ω	resistance
farad	<i>F</i>	electric capacitance
henry	<i>H</i>	electric inductance

Other units smaller or larger than the base unit are created by multiplying or dividing the base unit by powers of 10. The names of each of these smaller/larger units are identified by attaching a prefix to the name of the base unit. The first character or prefix indicates the power of 10 of the number of the base units being used.

Prefix	Symbol	Power of Ten	Number per Base Unit
nano	n	$10^{-9} = 0.000000001$	$10^9 = 1,000,000,000$
micro	μ	$10^{-6} = 0.000001$	$10^6 = 1,000,000$
milli	m	$10^{-3} = 0.001$	$10^3 = 1000$
centi	c	$10^{-2} = 0.01$	$10^2 = 100$
deci	d	$10^{-1} = 0.1$	$10^1 = 10$
deca	da	$10^1 = 10$	$10^{-1} = \frac{1}{10} = 0.1$
hecto	h	$10^2 = 100$	$10^{-2} = \frac{1}{100} = 0.01$
kilo	k	$10^3 = 1000$	$10^{-3} = \frac{1}{1000} = 0.001$
mega	M	$10^6 = 1,000,000$	$10^{-6} = \frac{1}{1,000,000}$
giga	G	$10^9 = 1,000,000,000$	$10^{-9} = \frac{1}{1,000,000,000} = 0.000000001$

Examples of Metric Units:

Centimeter (cm): Centimeters are a smaller unit than the meter. Centimeters are 10^{-2} or $\frac{1}{100}$ of a meter. This means there are 10^2 or 100 centimeters in one meter.

Kilohertz (kHz): Kilohertz are a larger unit than the hertz. A kilohertz is 10^3 or 1000 hertz. This means that there is 10^{-3} or $\frac{1}{1000}$ of a kilohertz in one hertz.

Milligrams (mg): Milligrams are a smaller unit than the gram. Milligrams are 10^{-3} of a gram. This means there are 10^3 or 1000 milligrams in one gram.

Conversions within the Metric System:

Conversions within the metric system simply require the shifting of the decimal point. As with conversions within the English System, we can use the idea of unit fractions to help us convert

between units in the Metric System. We create unit fractions by using what the prefix means in terms of the base unit.

Example: Convert 0.264 kilograms to grams.

Solution:
$$0.264 \text{ kilograms} = \frac{0.264 \cancel{\text{kg}}}{1} \times \frac{10^3 \text{ g}}{1 \cancel{\text{kg}}} = 264 \text{ grams}$$

Notice that in the example above, we shift the decimal point three places to the right since we are multiplying by 10^3 .

Example: Convert 175 millimeters to centimeters.

Solution:
$$175 \text{ mm} = \frac{175 \cancel{\text{mm}}}{1} \times \frac{1 \cancel{\text{m}}}{10^3 \cancel{\text{mm}}} \times \frac{10^2 \text{ cm}}{1 \cancel{\text{m}}} = \frac{175}{10} \text{ cm} = 17.5 \text{ cm}$$

Notice in this example, we end up shifting 1 decimal place to the left since we need to divide by 10.

Example: Convert 0.00059 kA to mA.

Solution:
$$0.00059 \text{ kA} = \frac{0.00059 \cancel{\text{kA}}}{1} \times \frac{10^3 \cancel{\text{A}}}{1 \cancel{\text{kA}}} \times \frac{10^3 \text{ mA}}{1 \cancel{\text{A}}} = 0.00059 \times 10^6 \text{ mA} = 590 \text{ mA}$$

Notice in this example, we end up shifting the decimal point 6 places to the right since we are multiplying by 10^6 .

Section 5.1.3: Performing Operations with Measurements

In many situations we may need to perform operations with measurements. We may need to add measurements to find the perimeter around an object; we may need to subtract to find out how much we have left; we may want to multiply measurements to find area or the amount of work required. In this section we take a look at how to perform these operations.

Section 5.1.3a: Addition and Subtraction of Measurements:

It is not sensible to add or subtract measurements of different kinds of things. For example, $15 \text{ lb} + 7 \text{ ft}$ is a meaningless operation. This is just the old adage that it's impossible to add apples and oranges! To add or subtract measurements requires the same kind of quantities, as in $9 \text{ feet} + 8 \text{ feet} = 17 \text{ feet}$. Note, we just add the numbers and carry the factor of the unit. This is just the distributive property discussed in the Algebra Chapter. In some problems there may be two different units involved in each measurement. In these problems the key is to add the like type of units.

Example: *Perform the indicated operation: 9 lb 8 oz + 4 lb 9 oz*

Solution:

$$\begin{array}{r} 9 \text{ lb } 8 \text{ oz} \\ + 4 \text{ lb } 9 \text{ oz} \\ \hline 13 \text{ lb } 17 \text{ oz} \end{array}$$

However since 17 oz is more than 1 pound's worth of ounces, we do not leave the answer in this form. We take 16 of the ounces and convert them to one pound. This leaves us with one ounce. We then add one pound to our current total number of pounds and we write the remaining number of ounces.

$$13 \text{ lb } 17 \text{ oz} = 13 \text{ lb} + 16 \text{ oz} + 1 \text{ oz} = 13 \text{ lb} + 1 \text{ lb } 1 \text{ oz} = 14 \text{ lb } 1 \text{ oz}$$

Our final answer is: 14 lb 1 oz.

What do we do, however, if the units of the measurements we are asked to add or subtract are not the same? We must do unit conversions so that both measurements have the same units.

Example: *Perform the indicated operation: 8 ft + 36 in.*

Solution: The quantities to be added are both lengths so the operation makes sense, but we can't actually perform the addition until we get the units to agree. We must either get both units to be inches or both to be feet.

Performing the indicated operation using feet:

We must first convert the 36 inches to feet as follows:

$$36 \text{ in} = \frac{36 \cancel{\text{in}}}{1} \times \frac{1 \text{ ft}}{12 \cancel{\text{in}}} = \frac{36}{12} \text{ ft} = 3 \text{ ft}$$

Now we perform the operation using 3 feet in place of the 36 inches.

$$8 \text{ ft} + 3 \text{ ft} = 11 \text{ ft}$$

Thus 8 ft + 36 in is 11 feet.

Performing the indicated operation using inches:

First we must convert 8 feet to inches as follows:

$$8 \text{ ft} = \frac{8 \cancel{\text{ft}}}{1} \times \frac{12 \text{ in}}{1 \cancel{\text{ft}}} = 96 \text{ in}$$

Now we perform the operation using 96 inches in place of 8 feet.

$$8 \text{ ft} + 36 \text{ in} = 96 \text{ in} + 36 \text{ in} = 132 \text{ in}$$

Thus an alternate answer to this problem is that 8 ft + 36 in is equal to 132 inches.

Example: *Perform the indicated operation: 4 ft – 14 in.*

Solution: First we convert the 4 feet to inches.

$$\frac{4 \cancel{\text{ft}}}{1} \times \frac{12 \text{ in}}{1 \cancel{\text{ft}}} = 48 \text{ in}$$

Next we replace the 4 ft by 48 in and do the subtraction.

$$4 \text{ ft} - 14 \text{ in} = 48 \text{ in} - 14 \text{ in} = 34 \text{ in}$$

We can write the answer in one of two ways. We can write the answer as 34 inches; however, it may be preferable to have the answer in terms of feet and inches. To convert this into feet and inches we divide 34 by 12 and we get 2 full feet (24 inches) and write the remaining number of inches.

$$\frac{34 \cancel{\text{in}}}{1} \times \frac{1 \text{ ft}}{12 \cancel{\text{in}}} = 2 \text{ ft } 10 \text{ in}$$

Thus an alternate answer to this problem is 2 feet and 10 inches.

Example: Perform the indicated operation: (5 tons 250 lb) – (2 ton 1225 lb)

Solution: If we set up the problem vertically as follows, we notice that we do not have enough pounds in the first measurement to do the subtraction.

$$\begin{array}{r} 5 \text{ tons } 250 \text{ lb} \\ -2 \text{ ton } 1225 \text{ lb} \\ \hline \end{array}$$

As a result, we must either borrow or convert all of the units to pounds and then do the subtraction. When borrowing, we borrow 1 from the tons column and convert it over to its equivalent value of 2000 lb. We add the 2000 lb to the number of pound we already have and do the subtraction.

$$\begin{array}{r} \cancel{5}^4 \text{ tons } \overset{2250}{\cancel{250} + 2000} \text{ lb} \\ -2 \text{ tons } \quad 1225 \text{ lb} \\ \hline 2 \text{ tons } \quad 1025 \text{ lb} \end{array}$$

Thus the answer is 2 tons 1025 lb.

Section 5.1.3b Multiplying and Dividing Measurements:

While adding or subtracting different kinds of measurements is impossible, multiplying or dividing measurements is always possible. As mentioned above there are two parts to every measurement. If we are asked to multiply/ divide two measurements we not only multiply/ divide the number parts of the measurements, we also multiply/ divide the unit parts of each measurement.

Example: Perform the indicated operation: 5 lb × 4 ft

Solution: $5 \text{ lb} \times 4 \text{ ft} = (5 \times 4)(\text{lb} \times \text{ft}) = 20 \text{ ft} \cdot \text{lb}$ where a $\text{ft} \cdot \text{lb}$ is 1 ft \times 1 lb is a “foot pound” which is a measure of either energy or torque.

Example: *Perform the indicated operation:* 100 miles \div 4 gallons

Solution: $100 \text{ miles} \div 4 \text{ gallons} = \frac{100 \text{ miles}}{4 \text{ gallons}} = 25 \text{ miles per gallon}$

Example: *Perform the indicated operation:* \$3.95 per gallon \div 32 miles per gallon

Solution: $\frac{\$3.95}{1 \text{ gallon}} \div \frac{32 \text{ miles}}{1 \text{ gallon}} = \frac{\$3.95}{1 \text{ gallon}} \times \frac{1 \text{ gallon}}{32 \text{ miles}} = \$0.1234375/\text{mile} \approx \$0.12/\text{mile}$

Example: *Perform the indicated operation:* $(12 \text{ in})^2$.

Solution: $(12 \text{ in})^2 = (12 \text{ in})(12 \text{ in}) = (12 \times 12)(\text{in} \times \text{in}) = 144 \text{ in}^2$

Your Turn!!

1. Convert 60 inches to feet. 1) _____

2. Convert 12 yards to feet. 2) _____

3. Convert 42,240 feet to miles. 3) _____

4. Convert 40 feet to yards and feet. 4) _____

5. Convert 5 feet 2 inches to inches. 5) _____

6. Convert 12,000 pounds to tons. 6) _____

7. Round to the nearest 32'nd of an inch: 0.165 in.

7) _____

8. Round to the nearest 64'th of an inch: 0.645 in

8) _____

9. What size bolt, to the nearest 64^{th} of an inch, will fit a hole 0.59 in diameter? 9) _____
10. How many micrograms are in 1 gram? 10) _____
11. How many joules are in a gigajoule? 11) _____
12. How many decimeters are in 1 meter? 12) _____
13. How many nanoseconds are in 1 second? 13) _____
14. Which is a larger unit: volt or kilovolt? 14) _____
15. Which is a larger unit: milliliter or centiliter? 15) _____
16. Convert 45.5 m to cm. 16) _____
17. Convert 40 mm to cm. 17) _____
18. Convert 355 m to km. 18) _____
19. Convert 1400 mm to meters. 19) _____
20. Convert 1585 cm to meters. 20) _____
21. Convert 8.3 cm to mm. 21) _____
22. Convert 525 grams to kilograms. 22) _____

23. Convert 4.5 grams to milligrams. 23) _____
24. Convert 48.2 mg to grams. 24) _____
25. Convert 6.3 grams to kilograms. 25) _____
26. Convert 4.01 kg to grams. 26) _____
27. Convert 4500 ml to Liters. 27) _____
28. Convert 410 volts to kilovolts. 28) _____
29. Convert 0.16 kilowatts to watts. 29) _____
30. Convert 3.6 liters to milliliters 30) _____
31. Convert 260 mA to A. 31) _____
32. Perform the indicated operation: $12\text{ ft } 7\text{ in} + 8\text{ ft } 8\text{ in}$
32) _____
33. Perform the indicated operation: $4\text{ yd} + 7\text{ ft}$
33) _____
34. Perform the indicated operation: $49\text{ in} - 2\text{ft}$
34) _____
35. Perform the indicated operation: $(29\text{ lb } 3\text{ oz}) - (8\text{ lb } 15\text{ oz})$
35) _____

36. Perform the indicated operation: $28 \text{ oz} + 3 \text{ gal}$

36) _____

37. Perform the indicated operation: $5 \text{ lb. } 6 \text{ oz.} - 2 \text{ lb. } 9 \text{ oz.}$

37) _____

38. Perform the indicated operation: $6 \text{ ton } 1540 \text{ lb} + 2 \text{ tons } 850 \text{ lb}$

38) _____

39. Perform the indicated operation: $4.2 \text{ cm} \times 3.5 \text{ cm}^2$

39) _____

40. Perform the indicated operation: $(12 \text{ in})^3$

40) _____

41. Perform the indicated operation: $(3 \text{ ft})^2$

41) _____

42. Perform the indicated operation: $465 \text{ miles} \div 23 \text{ mpg}$

42) _____

43. Perform the indicated operation: $590 \text{ miles} \div 56 \text{ mi/hr}$

43) _____

44. A 3.4 m rope is attached to a 5.8 m rope. However when the ropes are tied, 8 cm of length is lost to form the knot. What is the length of the tied ropes?

44) _____

45. The ice on Tiedmann's pond is 5.33 cm thick. For safe skating the town of Middleton requires 80 mm of ice thickness. How much thicker must the ice be in order for the pond to be available for skating?

45) _____

46. One bag of Pepperidge Farm Bordeaux cookies weighs $6\frac{3}{4}$ ounces. How many pounds will a dozen bags weigh?
46) _____
47. A can of 7-Up weighs 336 grams. Determine the weight in kilograms of a case of 24 cans.
47) _____
48. A floor tile is 22.86 cm wide. How many tiles will be in a row if the distance across the room is 3.429 m.
48) _____
49. Stanley paid \$14.00 to fill his car with 44.3 liters of gasoline. What is the price per liter of gasoline to the nearest cent?
49) _____
50. If the price per barrel of crude oil is \$127 and each barrel contains 31 gallons, how much per gallon is crude oil?
50) _____

Section 5.2: Measuring Area

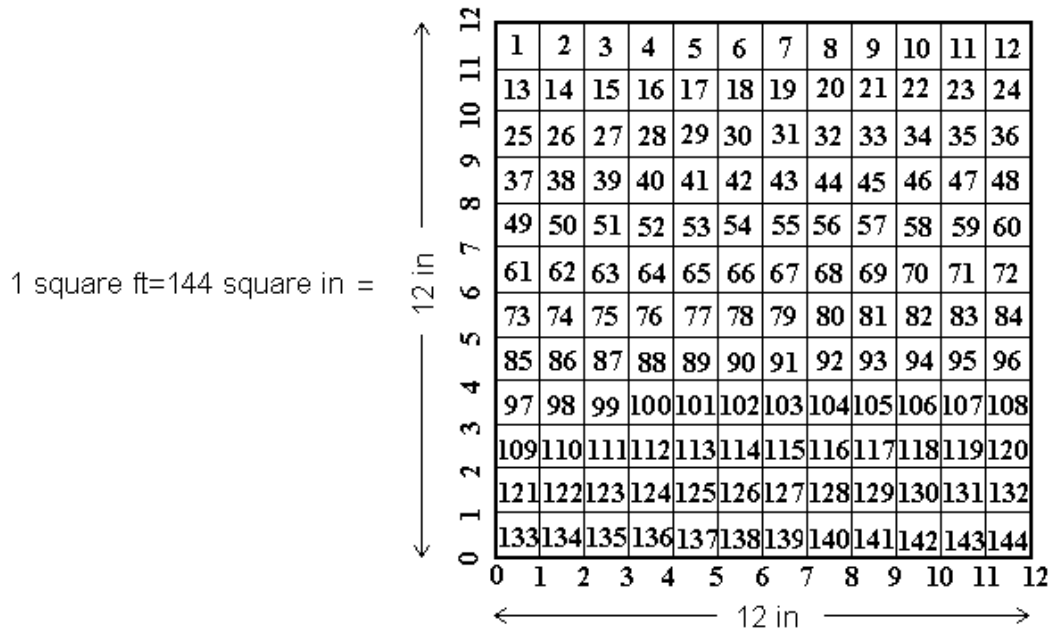
Up to this point we have looked at measurements that involve length, weight, and time. In many situations we need to be able to describe the size of a piece of cloth, of a piece of sheet metal, of a computer screen, of a wound, or of a piece of land. In this section we take a look at how to measure the amount of space inside a two-dimensional figure. The amount of “two-dimensional space” inside of a planar figure is called its **area**. How do we measure “two-dimensional space?” We use square units. A **square unit** is a square whose sides are each one unit in length. For example, a 2 ft by 6 ft rectangle has an area of $A = (\text{length})(\text{width}) = (2 \text{ ft})(6 \text{ ft}) = 12 \text{ ft}^2$. What does this mean? 12 one foot by one foot squares will fit inside this region.

Care must be taken when converting units of area. Suppose we want to calculate how many square inches are in an area of 1.6 ft^2 . In order to do this conversion, we need to know the number of square inches in a square foot. How is this determined?

$$1 \text{ ft}^2 = (1 \text{ ft})(1 \text{ ft}) = (12 \text{ in})^2 = 12^2 \text{ in}^2 = 144 \text{ in}^2$$

Note: when we evaluate $(12 \text{ in})^2$ we square **both** the number (12) **and** the units (in). This is illustrated below.

$$1 \text{ square inch} = \frac{1 \text{ in}}{1 \text{ in}} \square$$



Example: Convert 1.6 ft^2 to in^2 .

Solution: To perform the conversion we must multiply by the unit fraction $\frac{(12 \text{ in})^2}{(1 \text{ ft})^2}$ or $\frac{144 \text{ in}^2}{1 \text{ ft}^2}$.

$$1.6 \text{ ft}^2 = \frac{1.6 \text{ ft}^2}{1} \times \frac{(12 \text{ in})^2}{(1 \text{ ft})^2} = \frac{1.6 \cancel{\text{ft}^2}}{1} \times \frac{144 \text{ in}^2}{1 \cancel{\text{ft}^2}} = 230.4 \text{ in}^2$$

Thus there are 230.4 square inches in 1.6 square feet.

As mentioned above, we must be very careful when converting units of area. We must square the linear conversion factor to obtain the conversion factor for the square units used to measure area.

Example: Convert 35,000,000 ft^2 to square miles.

Solution:
$$35,000,000 \text{ ft}^2 = \frac{35,000,000 \text{ ft}^2}{1} \times \frac{(1 \text{ mi})^2}{(5280 \text{ ft})^2} = \frac{35,000,000 \text{ ft}^2}{1} \times \frac{1 \text{ mi}^2}{27,878,400 \text{ ft}^2}$$

$$= 1.25545225 \text{ mi}^2 \approx 1.3 \text{ mi}^2$$

There are 1.25545225 square miles or approximately 1.3 square miles in 35,000,000 square feet.

So far we have just looked at examples involving units from the English System of measurement. The same process works when working with units of area in the Metric System.

Example: Convert 0.042 m^2 to cm^2 .

Solution:
$$0.042 \text{ m}^2 = \frac{0.042 \text{ m}^2}{1} \times \frac{(10^2 \text{ cm})^2}{(1 \text{ m})^2} = \frac{0.042 \text{ m}^2}{1} \times \frac{10^4 \text{ cm}^2}{1 \text{ m}^2} = 420 \text{ cm}^2$$

Notice that the decimal point is shifted 4 places to the right since we are multiplying by 10^4 .

Let's turn to an application problem.

Example: A fire has spread to the entire floor of a building. If the floor of the building is 55 feet by 48 feet, how many square yards does the fire cover?

Solution: There are two methods for solving this problem. We can either convert each of the dimensions to yards before calculating the area, or we can calculate the area and then convert the square feet to square yards.

Solving by Calculating the Area First:

$$\text{Area} = (\text{length})(\text{width}) = (55 \text{ ft})(48 \text{ ft}) = 2640 \text{ ft}^2$$

Now we convert the square feet to square yards as follows:

$$2640 \text{ ft}^2 = \frac{2640 \text{ ft}^2}{1} \times \frac{(1 \text{ yd})^2}{(3 \text{ ft})^2} = \frac{2640 \cancel{\text{ft}^2}}{1} \times \frac{1 \text{ yd}^2}{9 \cancel{\text{ft}^2}} = \frac{2640}{9} \text{ yd}^2 = 293.\bar{3} \text{ yd}^2 \approx 293.3 \text{ yd}^2$$

The fire is covering $293\frac{1}{3} \text{ yd}^2$ or approximately 293.3 yd^2 .

Solving by Converting First:

First we convert each dimension as follows:

$$55 \text{ ft} = \frac{55 \cancel{\text{ft}}}{1} \times \frac{1 \text{ yd}}{3 \cancel{\text{ft}}} = \frac{55}{3} \text{ yd}$$

$$48 \text{ ft} = \frac{48 \cancel{\text{ft}}}{1} \times \frac{1 \text{ yd}}{3 \cancel{\text{ft}}} = \frac{48}{3} \text{ yd} = 16 \text{ yd}$$

Next we calculate the area:

$$\text{Area} = (\text{length})(\text{width}) = \left(\frac{55}{3} \text{ yd}\right)(16 \text{ yd}) = \frac{55 \times 16}{3} \text{ yd}^2 = \frac{880}{3} \text{ yd}^2 = 293\frac{1}{3} \text{ yd}^2$$

As found when calculating the area first, the fire is covering $293\frac{1}{3} \text{ yd}^2$ or approximately 293.3 yd^2 .

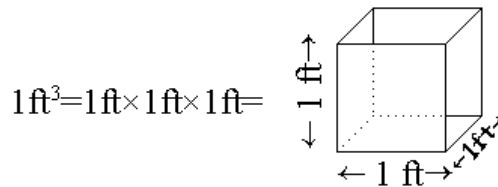
Your Turn!!

1. Convert 0.82 ft^2 to in^2 1) _____
2. Convert 329 in^2 to ft^2 . 2) _____
3. Convert 925 ft^2 to yd^2 . 3) _____
4. Convert 1.24 yd^2 to ft^2 . 4) _____
5. Convert 0.35 mi^2 to ft^2 . 5) _____
6. Convert $179,000,000 \text{ ft}^2$ to mi^2 . 6) _____
7. Convert 0.46 cm^2 to mm^2 . 7) _____
8. Convert 1239 cm^2 to m^2 . 8) _____
9. Convert $12,325 \text{ m}^2$ to km^2 . 9) _____

10. A piece of sheet metal is 36 inches by 27 inches. How many square feet of sheet metal are there?
10) _____
11. A wound is circular in shape. If the radius of the wound is 0.6 cm and area of a circle can be found by $\text{Area} = \pi(\text{radius})^2$ (where $\pi \approx 3.14$), approximate the area of the wound to the nearest square millimeter.
11) _____

Section 5.3: Measuring Volume

Just as we often want to measure the space inside a two-dimensional object, we have reason to measure the space inside a three-dimensional object. We may want to measure the space an engine takes up, the space inside a box, the amount of water we can fit inside a tank, or how much space we have for blood in a vial. **Volume** is the amount of “three-dimensional space” inside of a solid. How do we measure space inside a three-dimensional object? We use cubic units. A **cubic unit** is a cube whose length, width and height each have a length of 1 unit. For example, a 2 ft by 3 ft by 2 ft box has a volume, $V = (\text{length})(\text{width})(\text{height}) = (2 \text{ ft})(3 \text{ ft})(2 \text{ ft}) = 12 \text{ ft}^3$. Here the unit ft^3 is one cubic foot (cu ft), which literally means a one foot by one foot by one foot cube as shown below.



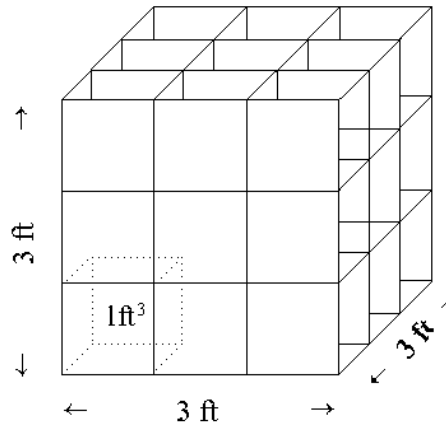
When we say that the volume of the box is 12 ft^3 , we mean that we could fit exactly 12 one foot by one foot by one foot cubes inside this box.

Like area conversions, volume conversions require careful setup. Suppose we wish to convert cubic feet to cubic yards. We must first determine the conversion factor between these units. To do this we use the same kind of technique as we did when determining the conversion factor between square units.

$$1 \text{ yd}^3 = (1 \text{ yd})(1 \text{ yd})(1 \text{ yd}) = (1 \text{ yd})^3 = (3 \text{ ft})^3 = 3^3 \text{ ft}^3 = 27 \text{ ft}^3$$

Note: when we evaluate $(3 \text{ ft})^3$ we cube **both** the number (3) **and** the units (ft) . This is illustrated below.

$$1 \text{ yd}^3 = 3 \text{ ft} \times 3 \text{ ft} \times 3 \text{ ft} = 27 \text{ ft}^3 =$$



Example: Convert 12 ft^3 to cubic yards.

$$\text{Solution: } 12 \text{ ft}^3 = \frac{12 \text{ ft}^3}{1} \times \frac{(1 \text{ yd})^3}{(3 \text{ ft})^3} = \frac{12 \cancel{\text{ft}^3}}{1} \times \frac{1 \text{ yd}^3}{27 \cancel{\text{ft}^3}} = \frac{12}{27} \text{ yd}^3 \approx 0.44 \text{ yd}^3$$

Example: Convert 2500 in^3 to ft^3 .

$$\begin{aligned} \text{Solution: } 2500 \text{ in}^3 &= \frac{2500 \text{ in}^3}{1} \times \frac{(1 \text{ ft})^3}{(12 \text{ in})^3} = \frac{2500 \cancel{\text{in}^3}}{1} \times \frac{1 \text{ ft}^3}{1728 \cancel{\text{in}^3}} \\ &= \frac{2500}{1728} \text{ ft}^3 = 1 \frac{193}{432} \text{ ft}^3 \approx 1.45 \text{ ft}^3 \end{aligned}$$

Thus 2500 cubic inches is about 1.45 cubic feet.

As before, the same process works for metric units as well.

Example: Convert 2695 mm^3 to cm^3 .

$$\text{Solution: } 2695 \text{ mm}^3 = \frac{2695 \text{ mm}^3}{1} \times \frac{(1 \text{ m})^3}{(10^3 \text{ mm})^3} \times \frac{(10^2 \text{ cm})^3}{(1 \text{ m})^3}$$

$$= \frac{2695 \cancel{\text{mm}^3}}{1} \times \frac{1 \cancel{\text{m}^3}}{10^3 \cancel{\text{mm}^3}} \times \frac{10^6 \text{cm}^3}{1 \cancel{\text{m}^3}} = 2.695 \text{ cm}^3$$

Notice that the decimal point is just shifted three places to the left since we are dividing by 10^3 .

Alternate Solution: If you take a look at a ruler, you will notice that there are 10 mm in each centimeter. Another method for solving this problem is to use this fact.

$$\begin{aligned} 2695 \text{ mm}^3 &= \frac{2695 \text{ mm}^3}{1} \times \frac{(1 \text{ cm})^3}{(10 \text{ mm})^3} = \frac{2695 \text{ mm}^3}{1} \times \frac{1 \text{ cm}^3}{10^3 \text{ mm}^3} \\ &= \frac{2695 \cancel{\text{mm}^3}}{1} \times \frac{1 \cancel{\text{cm}^3}}{1000 \cancel{\text{mm}^3}} = 2.695 \text{ cm}^3 \end{aligned}$$

As seen from both calculations 2695 mm^3 is equal to 2.695 cm^3 .

In some volume problems involving metric units, you may need to convert from cubic units to liters. The key here is to remember that the metric system was created so that $1 \text{ mL} = 1 \text{ cm}^3$. The key to doing conversions between cubic metric units and liters is to use a unit fraction involving this conversion factor.

Example: Convert 187 mm^3 to mL.

$$\begin{aligned} \text{Solution: } 187 \text{ mm}^3 &= \frac{187 \text{ mm}^3}{1} \times \frac{(1 \text{ m})^3}{(10^3 \text{ mm})^3} \times \frac{(10^2 \text{ cm})^3}{(1 \text{ m})^3} \times \frac{1 \text{ mL}}{1 \text{ cm}^3} \\ &= \frac{187 \cancel{\text{mm}^3}}{1} \times \frac{1 \cancel{\text{m}^3}}{10^3 \cancel{\text{mm}^3}} \times \frac{10^6 \cancel{\text{cm}^3}}{1 \cancel{\text{m}^3}} \times \frac{1 \text{ mL}}{1 \cancel{\text{cm}^3}} = 0.187 \text{ mL} \end{aligned}$$

Notice that we simply shifted the decimal point three places to the left since we were dividing by 10^3 . Thus 187 cubic millimeters is the same as 0.187 milliliters.

Lets turn to an application now.

Example: Determine the volume of a box that has dimensions 9 mm by 7.5 mm by 27 mm in cubic centimeters.

Solution: As with the area problem discussed earlier, there are two ways to do this problem. We can convert each of the dimensions to centimeters and then calculate the

4. Convert 0.78 yd^3 to ft^3 . 4) _____
5. Convert 8.75 gal to ft^3 . 5) _____
6. Convert 3.75 ft^3 to gallons. 6) _____
7. Convert 326 L to m^3 . 7) _____
8. Convert 3260 mm^3 to liters. 8) _____
9. Convert 3.5 m^3 to liters. 9) _____
10. Convert 56 mL to mm^3 . 10) _____
11. Convert 0.0575 m^3 to cm^3 . 11) _____
12. Convert 2345 mm^3 to cm^3 . 12) _____
13. Find the volume of a box in cubic yards that has dimensions 8 ft 3 in by 7 ft 9 in by 3 ft 6 in.
13) _____
14. Find the volume of a box in cubic centimeters that has dimensions 11 mm by 5.8 mm by 25 mm.
14) _____

Section 5.4: Conversion Between Metric and English Units

In the United States we often use the English System rather than the Metric System. However, we import and export products to countries that use the Metric System instead. This makes it important for us to be able to convert measurements between the two systems. Conversions between metric and English units require conversion factors. Because the systems are completely different, these conversions will involve decimal values. Depending on the level of accuracy required for our calculation we can use conversion factors that go to different numbers of decimal places. At the end of this chapter there is an appendix that lists conversion factors between the systems. Many of these will list eight decimal places of accuracy. If our calculations do not require this level of accuracy, we will often use a conversion factor that is rounded off to smaller number of decimal places.

How do we convert between units in the two systems? Just as we did when converting units within a system, we will use unit fractions to convert between units in the two systems.

Example: Convert 5.6 miles to kilometers.

Solution:
$$5.6 \text{ miles} = \frac{5.6 \cancel{\text{ miles}}}{1} \times \frac{1.609344 \text{ km}}{1 \cancel{\text{ mile}}} = \frac{5.6 \times 1.609344}{1} \text{ km}$$

$$= 9.0123264 \text{ km} \approx 9.0 \text{ km}$$

5.6 miles is about 9.0 kilometers.

Example: Convert 5.4 cm² to in².

Solution: As previously, we must be careful as we convert square units. In order to obtain the correct conversion factor, we must square the linear conversion factors.

$$5.4 \text{ cm}^2 = \frac{5.4 \text{ cm}^2}{1} \times \frac{(1 \text{ in})^2}{(2.54 \text{ cm})^2} = \frac{5.4 \cancel{\text{ cm}^2}}{1} \times \frac{1 \text{ in}^2}{6.4516 \cancel{\text{ cm}^2}}$$

$$= \frac{5.4}{6.4516} \text{ in}^2 = 0.837001674 \text{ in}^2 \approx 0.84 \text{ in}^2$$

Thus 5.4 cm² is about 0.84 in².

Example: Convert 1.80 gallons per minute to m³ per hour.

Solution:
$$1.80 \text{ gal/min} = \frac{1.80 \cancel{\text{ gal}}}{1 \cancel{\text{ min}}} \times \frac{3.7854 \cancel{\text{ L}}}{1 \cancel{\text{ gal}}} \times \frac{10^{-3} \text{ m}^3}{1 \cancel{\text{ L}}} \times \frac{60 \cancel{\text{ min}}}{1 \text{ hour}}$$

$$= \frac{1.80 \times 3.7854 \times 10^{-3} \times 60 \text{ m}^3}{1 \text{ L}} = 0.4088232 \text{ m}^3/\text{hr} \approx 0.409 \text{ m}^3/\text{hr}$$

Thus 1.80 gallons per minute is about 0.409 cubic meters per hour.

The conversion between temperatures in the two systems requires the use of a formula rather than the use of unit fractions and conversion factors. To convert from a centigrade temperature to a Fahrenheit temperature, we use the formula: $\text{Temp}^{\circ}\text{F} = \text{Temp}^{\circ}\text{C} \times \frac{9}{5} + 32^{\circ}\text{F}$.

Example: Convert 40°C to degrees Fahrenheit.

Solution: $\text{Temp}^{\circ}\text{F} = \cancel{40}^{\cancel{8}} \times \frac{9}{\cancel{5}} + 32^{\circ}\text{F} = (72 + 32)^{\circ}\text{F} = 104^{\circ}\text{F}$

To convert from a Fahrenheit temperature to a centigrade temperature, we use the formula

$$\text{Temp}^{\circ}\text{C} = (\text{Temp}^{\circ}\text{F} - 32) \times \frac{5}{9}.$$

Example: Convert 10°F to degrees Celsius.

Solution: $\text{Temp}^{\circ}\text{C} = (10 - 32) \times \frac{5}{9} = (-22) \frac{5}{9} = \frac{-22 \times 5}{9} = -12 \frac{2}{9} \approx -12.2$

Thus 10°F is equivalent to about -12.2°C .

Temperature is a measure of heat. As a result, some scientists do not find having negative temperatures meaningful in their work. A third scale of measuring temperature, measured in degrees Kelvin, is built using the centigrade scale but it eliminates negative temperatures. In this system 0 degrees is at absolute zero. Absolute zero is the coldest possible temperature; the temperature at which molecules stop moving. To convert from degrees Celsius to degrees Kelvin we simply use the formula: $\text{Temp}^{\circ}\text{K} = \text{Temp}^{\circ}\text{C} + 273.15$.

Example: Convert 12.5°C to degrees Kelvin.

Solution: $\text{Temp}^{\circ}\text{K} = 12.5 + 273.15 = 285.65$

Thus 12.5°C is the same as 285.65°K

To convert from degrees Kelvin to degrees Celsius we simply use the formula:

$$\text{Temp}^{\circ}\text{C} = \text{Temp}^{\circ}\text{K} - 273.15$$

Example: Convert 25.8°K to degrees Celsius.

Solution: $\text{Temp}^{\circ}\text{C} = 25.8 - 273.15 = -247.35$

Thus 25.8°K is the same as -247.35°C .

Your Turn!!

1. Convert 7 ft to meters. 1) _____
2. Convert 9 inches to centimeters. 2) _____
3. Convert 14 meters to yards. 3) _____
4. Convert 26.5 yards to meters. 4) _____
5. Convert 15 km to miles. 5) _____
6. Convert 82 miles to km. 6) _____
7. Convert 25 m to feet. 7) _____
8. Convert 17.5 cm to inches. 8) _____
9. Convert 5 gal to Liters. 9) _____
10. Convert 280 L to gallons. 10) _____
11. Convert 23 qt to liters. 11) _____
12. Convert 19 liters to quarts. 12) _____
13. Convert 2.49 gallons to milliliters. 13) _____
14. Convert 4.5 liters to ounces. 14) _____

15. Convert 32 lb to kg. 15) _____
16. Convert 7 oz to grams. 16) _____
17. Convert 16 kg to lb. 17) _____
18. Convert 126 g to ounces. 18) _____
19. Convert 55 km/hr to mi/hr. 19) _____
20. Convert 62 mi/hr to km/hr. 20) _____
21. Convert 83 mph (meters per hour) to ft/min. 21) _____
22. Convert 2.56 square feet to cm^2 . 22) _____
23. Convert 179 cm^3 to in^3 . 23) _____
24. Convert 465 ft^2 to m^2 . 24) _____
25. Convert 3.2 m^3 to ft^3 . 25) _____
26. Convert 19.5 gal/min to ft^3/hour . 26) _____
27. Convert 25.8 mi/gal to km/L. 27) _____
28. Convert 35°C to $^\circ\text{F}$. 28) _____

29. Convert 98°F to $^{\circ}\text{C}$. 29) _____

30. Convert -23.5°F to $^{\circ}\text{C}$. 30) _____

31. Convert -5.75°C to $^{\circ}\text{F}$. 31) _____

32. Convert 47.5°K to $^{\circ}\text{C}$. 32) _____

33. Convert -125.5°C to $^{\circ}\text{K}$. 33) _____

34. Convert -17.5°C to $^{\circ}\text{K}$ 34) _____

35. Convert 365°K to $^{\circ}\text{C}$. 35) _____

36. Convert -36.5°F to $^{\circ}\text{K}$. 36) _____

37. Convert 46°F to $^{\circ}\text{K}$. 37) _____

38. Convert 65.4°K to $^{\circ}\text{F}$. 38) _____

39. Convert 301.6°K to $^{\circ}\text{F}$. 39) _____

40. A wire that is 13 mm wide is how many inches wide?
40) _____

41. The Weston's left Seattle on a cruise and were told that they would travel 76 miles before they entered Canadian water. At that point they would travel an additional 157 km until their final destination. How many kilometers was the entire trip?
41) _____

42. Ralph got a deal on a car that he bought used in the Bahamas. The only problem was the speedometer was marked only in kilometers per hour (km/hr). While driving through the state of Texas he was pulled over for traveling too slowly in the express lane of an interstate. He told the officer that he was doing 65, but this was km/hr. How fast was he going in MPH, miles per hour?
- 42) _____
43. How large would a farm of 320 acres be if it were measured in square kilometers?
- 43) _____
44. A box has dimensions of 5 ft 6 in by 3 ft 9 in by 3 ft 4 in. How many cubic meters is this?
- 44) _____
45. A car has a gas tank with a capacity of 50 L. If the car gets 33.5 miles per gallon, how many km can the car travel on a full tank?
- 45) _____

Chapter 5 Practice Exam

1. Perform the following calculation with measurement numbers: 15 yd. 1ft. – 9 yd. 2ft.
2. Perform the following calculation with measurement numbers: 1.6 MV + 14.4 MV
3. Perform the following calculation with measurement numbers: $4.75 \text{ in}^2 \times 3.6 \text{ in}$
4. Perform the following calculation with measurement numbers: 425 km \div 95 km per hour
5. Convert 56°C to $^\circ\text{F}$.
6. Convert 14.5 liters to gallons.
7. Convert 18.5 lbs. to kg.

8. Convert 976 ft^3 to in^3 .
9. Convert 36.5 mA to A .
10. Convert 0.945 m to cm .
11. Convert 75 m/sec to km/h .
12. Convert 54.5 yd^2 to in^2 .
13. Convert 21.65 ft^3 to gal .
14. Convert 4.25 L to in^3 .
15. Convert 26.5 in^3 to cm^3 .
16. What size bolt, to the nearest 64^{th} of an inch, will fit a hole $0.365''$ in diameter?
17. Joe's Service station has a drum that has 40 gallons and 3 quarts of oil left in it. How many oil changes can they perform if each oil change requires 6 quarts of oil?
18. To pour a concrete driveway you must order the concrete in cubic yards. If the driveway is to be 10 feet wide, 75 feet long and 4 inches thick, how many cubic yards of concrete are needed? Round your answer to one decimal place.
19. A 2.95 meter long sash cord has become frayed at both ends, so 1.25 cm is trimmed from each end. How long is the remaining cord?
20. A car has a gas tank with a capacity of 46 L. If the car gets 54.5 kilometers per gallon, how many miles can the car travel on a full tank?

Chapter 5 Appendix: Conversion Relations

Unit	Abbreviation	What it Measures
Inch	in	length
Feet	ft	length
Yard	yd	length
Mile	mi	length
Ounce	oz	volume
Pint	pt	volume
Quart	qt	volume
Gallon	gal	volume
Pound	lb	weight
degrees Fahrenheit	$^{\circ}F$	temperature
square inches	in^2	area
square feet	ft^2	area
square yards	yd^2	area
square miles	mi^2	area

Linear Measure:

1 ft = 12 in
 1 yd = 3 ft
 1 mile = 5280 ft
 1 rod = 16.5 ft
 1 furlong = 220 yd
 1 in = 2.54 cm
 1 ft = 0.3048 m
 1 yd = 0.9144 m
 1 mile = 1.609344 km

Weight Measure:

16 oz = 1 lb
 1 oz = 28.348 g
 1 ton = 2000 lb
 1 lb = 453.568 g
 1 kg = 2.20474 lb

Area Measure:

1 acre = 160 sq rods
 1 sq mile = 640 acres
 $1 \text{ cm}^2 = 0.15500031 \text{ in}^2$
 $1 \text{ m}^2 = 1.195990046 \text{ yd}^2$
 $1 \text{ km}^2 = 0.3861021585 \text{ sq mile}$

Time Measure:

1 min = 60 s
 1 hr = 60 min
 1 hr = 3600 s
 1 day = 24 hours
 1 yr = 365 days

Volume Measure:

16 oz = 1 pt
 2 pt = 1 qt
 4 qt = 1 gal
 $1 \text{ gal} = 0.13368056 \text{ ft}^3$
 $1 \text{ gal} = 231 \text{ in}^3$
 $1 \text{ gal} = 3.78541178 \text{ L}$
 $1 \text{ ft}^3 = 7.48051948 \text{ gal}$
 $1 \text{ ft}^3 = 28.31684659 \text{ L}$
 $1 \text{ L} = 0.26417205 \text{ gal}$
 $1 \text{ L} = 1.056688209 \text{ qt}$
 $1 \text{ L} = 61.02374409 \text{ in}^3$
 $1 \text{ L} = 0.001 \text{ m}^3$
 $1 \text{ mL} = 1 \text{ cm}^3$

Temperature Conversions:

$$\text{Temp}^{\circ}\text{F} = \text{Temp}^{\circ}\text{C} \times \frac{9}{5} + 32^{\circ}\text{F}$$

$$\text{Temp}^{\circ}\text{K} = \text{Temp}^{\circ}\text{C} + 273.15$$

$$\text{Temp}^{\circ}\text{C} = (\text{Temp}^{\circ}\text{F} - 32) \times \frac{5}{9}$$

$$\text{Temp}^{\circ}\text{C} = \text{Temp}^{\circ}\text{K} - 273.15$$

Chapter 5 Answer Section

Section 5.1 Solution Set:

1. 5 feet; 2. 36 feet; 3. 8 miles; 4. 13 yd 1 ft; 5. 62 inches; 6. 6 tons; 7. $\frac{5}{32}$ inch; 8. $\frac{41}{64}$ inch;
 9. $\frac{37}{64}$ inch; 10. 10^6 or 1,000,000 mg; 11. 10^9 or 1,000,000,000 joules; 12. 10 decimeters;
 13. 10^9 or 1,000,000,000 nanoseconds; 14. kilovolt; 15. centiliter; 16. 4550 cm; 17. 4 cm;
 18. 0.355 km; 19. 1.4 m; 20. 15.85 m; 21. 83 mm; 22. 0.525 kg; 23. 4500 mg; 24. 0.0482 g;
 25. 0.0063 kg; 26. 4010 g; 27. 4.5 L; 28. 0.41 kV; 29. 160 w; 30. 3600 ml; 31. 2.6 A;
 32. 21 ft 3 in; 33. 19 ft or $6\frac{1}{3}$ yd or 6 yd 1 ft; 34. 25 in or $2\frac{1}{12}$ ft or 2 ft 1 in; 35. 20 lb 4 oz;
 36. 384 oz or $3\frac{7}{32}$ gal; 37. 2 lb 13 oz; 38. 9 tons 390 lb; 39. 14.7 cm^3 ; 40. 1728 in^3 ; 41. 9 ft^2
 42. ≈ 20.2 gal; 43. ≈ 10.5 hours; 44. 9.12 m; 45. 26.7 mm or 2.67 cm; 46. 5.0625 lb or $5\frac{1}{16}$ lb;
 47. 8.064 kg; 48. 15 tiles; 49. $\approx \$0.316/\text{liter}$; 50. $\approx \$4.097/\text{gallon}$

Section 5.2 Solution Set:

1. 118.08 in^2 ; 2. $2\frac{41}{144} \text{ ft}^2 \approx 2.28 \text{ ft}^2$; 3. $102\frac{7}{9} \text{ yd}^2 \approx 102.8 \text{ yd}^2$; 4. 11.2 ft^2 ; 5. $9,757,440 \text{ ft}^2$;
 6. 6.42 m^2 ; 7. 46 mm^2 ; 8. 0.1239 m^2 ; 9. 0.012325 km^2 ; 10. 6.75 ft^2 ; 11. 113 mm^2

Section 5.3 Solution Set:

1. 5.327 ft^3 ; 2. 65.7 in^3 ; 3. 15.8 yd^3 ; 4. 21.1 ft^3 ; 5. 1.17 ft^3 ; 6. 28.1 gal; 7. 0.326 m^3 ;
 8. 0.00326 L; 9. 3500 L; 10. $56,000 \text{ mm}^3$; 11. $57,500 \text{ cm}^3$; 12. 2.345 cm^3 ; 13. 8.29 yd^3 ;
 14. 1.595 cm^3

Section 5.4 Solution Set:

1. 2.13 m; 2. 22.86 cm; 3. 15.3 yd; 4. 24.2 m; 5. 9.3 mi; 6. 132 km; 7. 82 ft; 8. 6.89 in;
 9. 18.9 L; 10. 74 gal; 11. 21.8 L; 12. 20.1 qt; 13. 9426 mL; 14. 152.2 ounces; 15. 14.5 kg;
 16. 198.4 g; 17. 35.3 lb; 18. 4.44 ounces; 19. 34.2 mi/hr; 20. 99.8 km/hr; 21. 4.54 ft/min;
 22. 2378.3 cm^2 ; 23. 10.9 in^3 ; 24. 43.2 m^2 ; 25. 113 ft^3 ; 26. $156.4 \text{ ft}^3/\text{hr}$; 27. 10.97 km/L ;
 28. 95°F ; 29. $36\frac{2}{3}^\circ\text{C} \approx 36.7^\circ\text{C}$; 30. $-30\frac{5}{6}^\circ\text{C} \approx -30.83^\circ\text{C}$; 31. 21.65°F ; 32. -225.65°C ;
 33. 147.65°K ; 34. 255.65°K ; 35. 91.85°C ; 36. 235.1°K ; 37. 280.9°K ; 38. -341.95°F ;
 39. 83.21°F ; 40. 0.51 in; 41. 279.3 km; 42. 40.4 MPH; 43. 1.29 km^2 ; 44. 1.95 m^3 ; 45. 712 km

Chapter 5 Practice Exam Solution set:

- 1.** 5 yd 2 ft; **2.** 16 MV; **3.** 17.1 in^3 ; **4.** 4.5 hours; **5.** 132.8°F ; **6.** 3.83 gal; **7.** 8.39 kg;
8. $1,686,528 \text{ in}^3$; **9.** 0.0365 A; **10.** 94.5 cm; **11.** 270 km/hr; **12.** $70,632 \text{ in}^2$; **13.** 162.0 gal;
14. 259.4 in^3 ; **15.** 434.3 cm^3 ; **16.** $\frac{23}{64} \text{ in}$; **17.** 27 oil changes; **18.** 9.26 yd^3 ; **19.** 293.75 cm;
20. 411.5 mi

Chapter 6 – Geometry

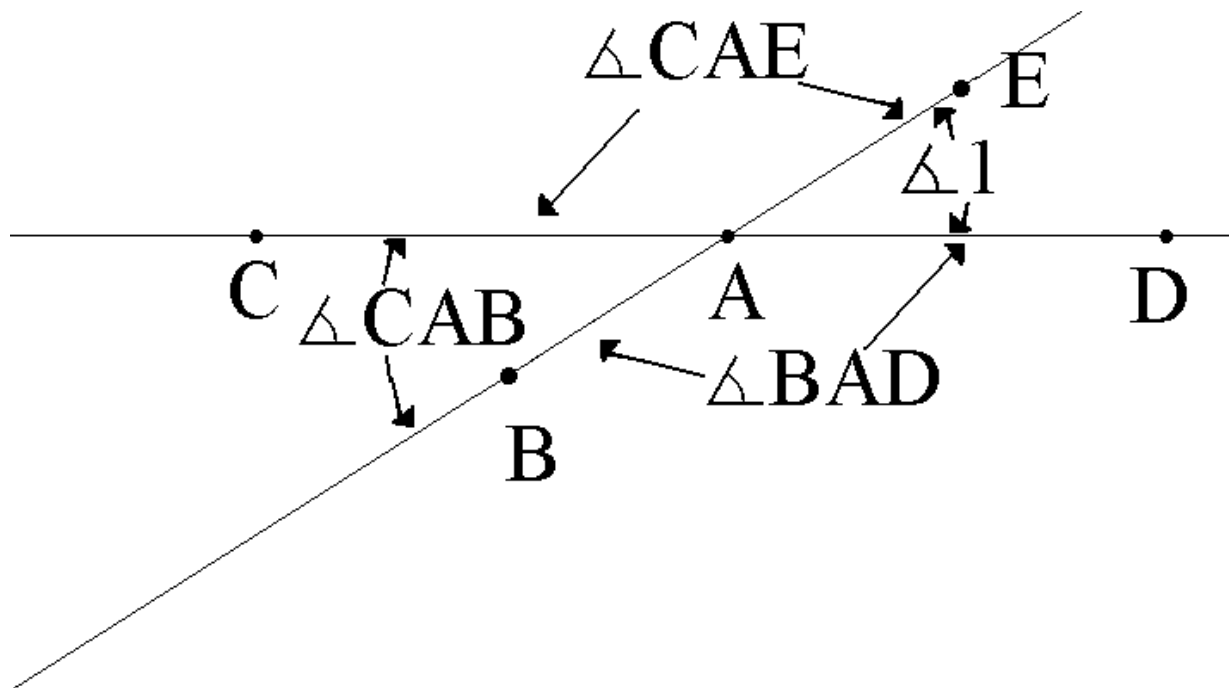
Geometry is a word from Greek which combines the word *geo* meaning *Earth* in Greek and the word *metry* which means to *measure* . So geometry literally means measuring the Earth. More generally geometry is the study of shapes, their properties, and how to measure them.

Section 6.1 Plane Geometry

In this section we are considering plane or flat two-dimensional geometry. The fundamental notions used to describe planar objects are points and lines.

Basic Facts About Points and Lines:

- Two different points define one and only one line that passes through both of them
- We shall use capital letters such as **A** to label points and a pair of letters such as **AB** to label the segment of the line between **A** and **B**.
- Line segments are measured in units of length such as feet or meters.
- The shortest distance between the two points is along this line.
- Two lines in a plane either meet or in other words intersect at a single point or are parallel, meaning that the two lines never touch.
- Where the lines meet four “openings” or angles are formed as shown below. The symbol \angle is used for the word angle. (See figure below)



Angles

In order to discuss angles, several technical terms are used. As shown in the above picture, the point where the lines intersect **A** is called the **vertex** of the four angles.

The separate angles are then named by indicating a point on each of the two lines which act as sides of the angle. Thus $\angle \mathbf{BAD}$ is the angle with vertex at **A** and with sides that include segments **AB** and **AD**. This same angle could also be labeled as $\angle \mathbf{DAB}$. A less precise but more convenient notation is to label angles by a single letter or number, provided that the symbol is placed appropriately in the diagram. For example, in the above diagram $\angle \mathbf{1} = \angle \mathbf{EAD}$.

Measuring Angles

The most commonly used measure for angles is the degree system. In the degree system we divide a circle into 360 equal units of angle called a degree and is given by the symbol $^{\circ}$. For precise surveying or problems in astronomy a smaller unit is required. This leads to the minute, a $1/60^{\text{th}}$ of a degree (symbolized by $'$), and the second, a $1/60^{\text{th}}$ of a minute (symbolized by $''$).

Thus, we have $1' = \frac{1}{60}^{\circ}$ and $1'' = \frac{1}{60}' = \frac{1}{60} \times \frac{1}{60}^{\circ} = \frac{1}{3600}^{\circ}$. An angle to the nearest second could be specified as $\angle 4 = 37^{\circ}29'54''$, this notation is called DMS for Degree Minute Second. This same angle could be given in DD (Decimal Degrees) as $\angle 4 = (37 + \frac{29}{60} + \frac{54}{3600})^{\circ} = 37.4983333...^{\circ}$. To go

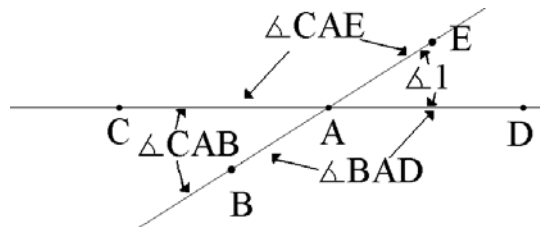
back to DMS, $\angle 4 = 37.4983333...^{\circ} = 37^{\circ} + \frac{60'}{1^{\circ}} \times .498333...^{\circ} = 37^{\circ}29.9' = 37^{\circ}29' + .9' \times \frac{60''}{1'} = 37^{\circ}29'54''$.

These conversions have been built into many calculators. Since each of you may have different calculators, *please ask your instructor for help if you're not sure how to use your calculator.*

Facts about measuring angles

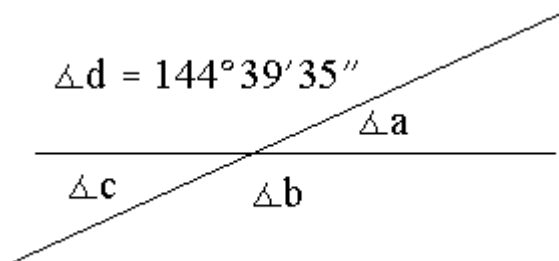
- Since there are 360° in a circle, the angular measure of a straight line is 180°
- In a square or right angle there are 90° .
- Two angles whose sum is 180° make a straight line and are called **supplementary**.

If two lines intersect as shown below: $\angle \mathbf{EAD} + \angle \mathbf{CAE} = \angle \mathbf{EAD} + \angle \mathbf{BAD} = 180^{\circ}$ and $\angle \mathbf{EAD} + \angle \mathbf{CAE} = \angle \mathbf{EAD} + \angle \mathbf{BAD} = 180^{\circ}$. Since $\angle \mathbf{EAD}$ is on both sides of this equation, we conclude that $\angle \mathbf{CAE} = \angle \mathbf{BAD}$ and similarly that $\angle \mathbf{CAB} = \angle \mathbf{EAD}$. These equal angles formed by two intersecting lines are called **vertex (or vertical) angles**.



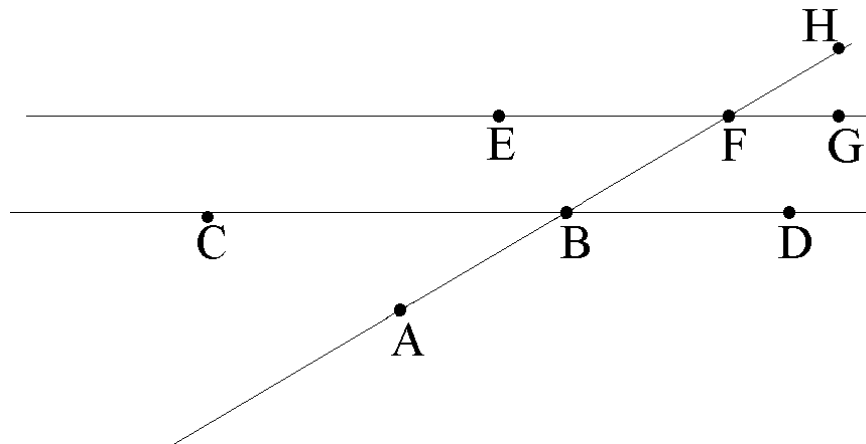
EXAMPLE:

Consider the following diagram:



SOLUTION: Once the measure of $\angle d$ is known, the remaining three can be determined. Since they are vertex angles, $\angle b = \angle d = 144^\circ 39' 35''$. Since $\angle a$ is the supplement to $\angle d$, $\angle a = 180^\circ - 144^\circ 39' 35''$. So $144^\circ 39' 35'' = 179^\circ 59' 60'' - 144^\circ 39' 35''$. So $180^\circ - 144^\circ 39' 35'' = 35^\circ 20' 25'' = \angle c$. This same calculation can also be done on a on most scientific calculators using a few keystrokes.

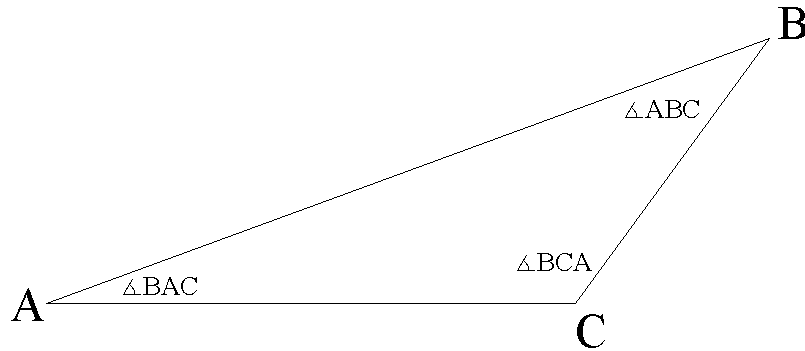
Consider a pair of parallel lines crossed by a third line (called the **transversal**) as shown below. If we imagine that point **B** is superimposed, or in other words, moved so that it is directly over point **F** by moving segment **CD** onto the line through **EG**, the **corresponding** angles $\angle EFB$ and $\angle CBA$ are equal. Similarly, $\angle EFH = \angle CBF$, $\angle ABD = \angle BFG$, and $\angle FBD = \angle HFG$. From the equality of vertex angles, the **alternating interior** (alternate sides of the transversal, inside the two parallel lines) are equal, i.e., $\angle EFB = \angle FBD$ and $\angle CBF = \angle BFG$. Similarly, the **alternating exterior** (alternate sides of the transversal, outside the two parallel lines) are equal, i.e., $\angle EFH = \angle ABD$ and $\angle CBA = \angle HFG$.

**Polygons**

Polygons are closed figures in the plane whose sides are line segments.

Triangles

The simplest polygon is the three-sided **triangle**. The points at the corners **A**, **B**, and **C** are the **vertices** of the triangle, and the angles $\angle \mathbf{BAC}$, $\angle \mathbf{BCA}$, and $\angle \mathbf{ABC}$ are called the interior angles of the triangle. The triangle is often then labeled as triangle **ABC**.

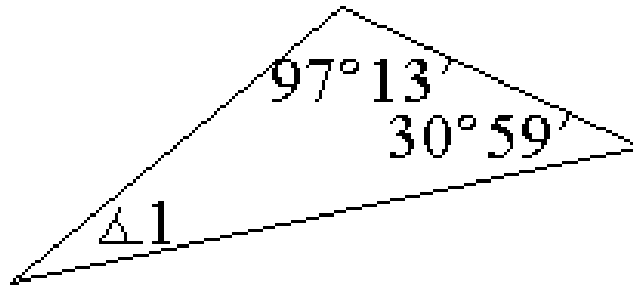


Important Fact:

The interior angles of a triangle always add up to to 180° .

EXAMPLE: In the triangle below the missing angle $\angle 1$ is calculated as follows :

SOLUTION: $\angle 1 = 180^\circ - 97^\circ 13' - 30^\circ 59' = 51^\circ 48'$



What else do we know about triangles? The first question we need to consider is what determines a triangle. Every triangle has three sides and three angles, so six numbers (three angle measures and three side lengths) are associated with every triangle. Two triangles are called **congruent** if one can be superimposed (or in other words placed on top of the other) so that the triangles fit exactly on the other. In other words, the triangles have exactly the same shape and size.

Essentially, triangles that are congruent are the “same”. There are three rules for congruency when using triangles:

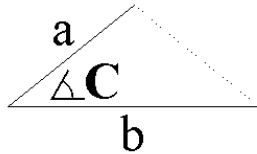
**Side Angle Side (abbreviated SAS) ,
Angle Side Angle (abbreviated ASA) , and
Side Side Side (abbreviated SSS) .**

The notation **SAS** means the length of two sides and the angle between these two sides are known; the remaining side and two angles are to be found out. The notation **ASA** (or equivalently **AAS**, since if any two angles in a triangle are known, the third can be determined from 180° minus the sum of the other two) means two angles and the side between them have been specified. Of course, the sum of the two specified angles must be less than 180° . Finally, **SSS** means all three sides have been specified. Since the shortest distance between any two vertices is the side joining these points, the longest of the three sides of a triangle must be shorter than the sum of the other two sides for the triangle to exist.

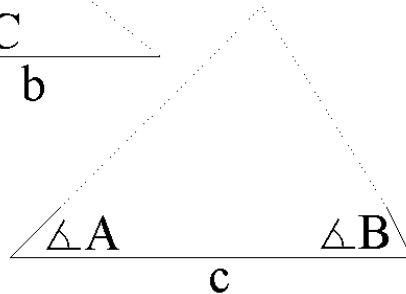
Note: AAA , (meaning that all three angles of the triangle are known, but that the lengths of all of the sides of the triangle are not known) determines a triangle's shape, but does NOT determine its size, since there could be two triangles with the same shape but be different in size, and is therefore **not** a rule of congruence.

Rules of Congruence

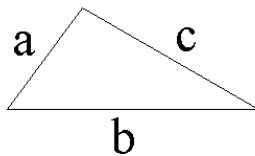
1. SAS



2. ASA



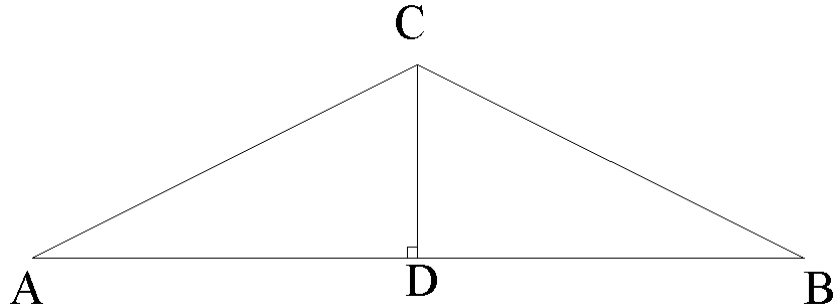
3. SSS



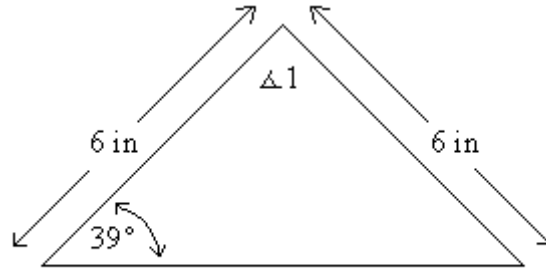
Types of Triangles

A triangle is called **isosceles** if two sides are equal. Consider the isosceles triangle **ACB** with **AC = CB** , from **C** construct the segment **CD** to **D** the mid point (i.e., **AD = DB**) of **AB** . Now by SSS triangle **ADC** is congruent to triangle **BDC** . Thus, $\angle ADC = \angle BDC$, and since these two angles sum to 180° , **CD** is perpendicular (makes a right angle) to **AB** . This is indicated in

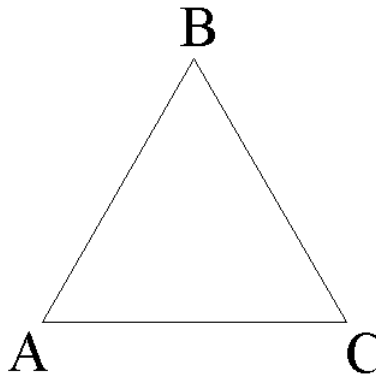
the diagram by the “little” box at **D** . Also $\angle CAB = \angle CBA$; in words, the angles opposite to the equal sides are also equal.



Consider the following isosceles triangle with $\angle 1$ unspecified. Since the angles opposite the equal sides must be equal, $\angle 1 = 180^\circ - 2(39^\circ) = 180^\circ - 78^\circ = 102^\circ$.



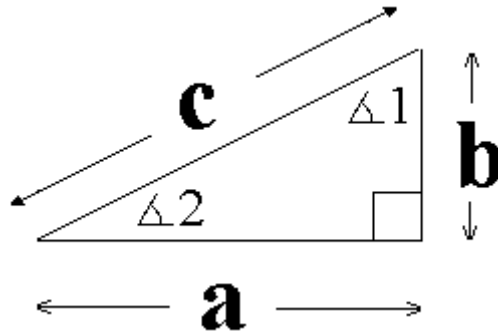
A triangle is called **equilateral** if all three sides are equal.



Of course, equilateral triangles are also isosceles, so $AB = BC$ and the opposite angles $\angle BAC$ and $\angle BCA$ are equal. But $AC = BC$, so $\angle ABC$ and $\angle BAC$ are equal . Thus all three angles are equal and since they sum to 180° , we have that for an equilateral triangle all of the internal angles equal 60° .

A **right triangle** is a triangle with a 90° interior angle. The large side opposite the 90° angle is called the **hypotenuse** while the remaining two sides are called **legs**.

In the diagram shown, the hypotenuse is **c** and the legs are **a** and **b**. The two **acute** (less than 90°) angles are $\angle 1$ and $\angle 2$ with $\angle 1$ opposite to the side of length **a** and $\angle 2$ opposite to the side of length **b**. The sum of the three interior angles is $\angle 1 + \angle 2 + 90^\circ = 180^\circ$, so we have the result that the two acute angles in a right triangle are **complementary**, i.e., $\angle 1 + \angle 2 = 90^\circ$.



All **right** triangles have a special relationship between the sides. This result is called the **Pythagorean Theorem** and is probably the most famous and useful result in geometry! The result can be stated a number of different ways. To calculate the hypotenuse knowing the lengths of the legs, we take the square root and get

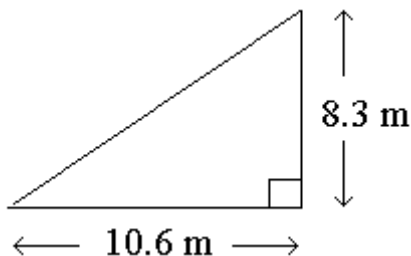
$$c = \sqrt{a^2 + b^2}.$$

To calculate a leg, say **a**, knowing the hypotenuse and the other leg, **b**, rearrange the formula to

$$a = \sqrt{c^2 - b^2} \quad \text{or} \quad \text{leg} = \sqrt{\text{hypotenuse}^2 - \text{other leg}^2}.$$

Remember the square root symbol is also a grouping symbol. Parentheses need to be used!

EXAMPLE: Find the hypotenuse of the right triangle pictured below.



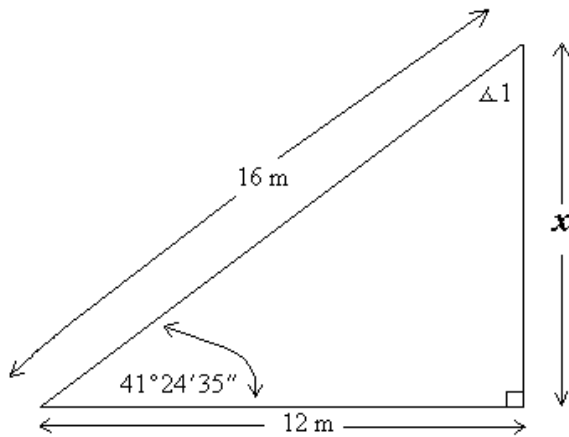
To find the missing hypotenuse of the following right triangle, we compute as follows:

$$c = \sqrt{\left((8.3\text{m})^2 + (10.6\text{m})^2\right)} = \sqrt{68.89\text{m}^2 + 112.36\text{m}^2} = \sqrt{181.25\text{m}^2} = 13.5 \text{ m}$$

Note: The implied parenthesis inside the square root has been made explicit as required to get the correct answer on the calculator. The units work out to be linear units as required for a length since $\sqrt{\text{m}^2} = \text{m}$.

EXAMPLE: consider finding the unknown quantities in the following right triangle. Find the length of the missing leg x .

SOLUTION: The missing dimension x is a leg and can be calculated from

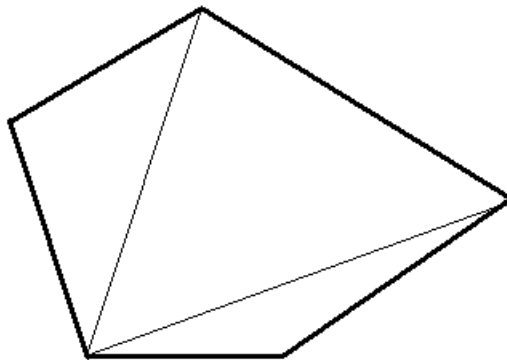


$$x = \sqrt{\left((16\text{m})^2 - (12\text{m})^2\right)} = \sqrt{112\text{m}^2} = 10.6\text{m} .$$

The missing angle is complementary to $41^\circ 24' 35''$, so

$$\angle 1 = 90^\circ - 41^\circ 24' 35'' = 48^\circ 35' 25'' .$$

Other Polygons (Polygons with more than 3 sides)



Consider the five-sided pentagon above. By drawing three triangles from one of the vertices, we see that the sum of the internal angles of the pentagon is the sum of all the internal angles in the three triangles or $3(180^\circ) = 540^\circ$. A similar argument for an n -sided polygon shows that the sum of the internal angles is $(n-2)(180^\circ)$.

EXAMPLE:

What do you get when you add up all of the interior angles in a 7 sided polygon?

SOLUTION: In this case $n = 7$, so $(n-2)(180^\circ) = (7-2)(180^\circ) = 5(180^\circ) = 900^\circ$.

EXAMPLE:

What do you get when you add up all of the interior angles in a 100 sided polygon?

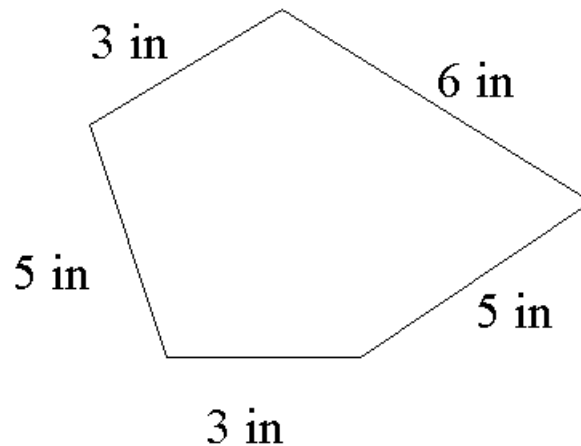
SOLUTION: In this case $n = 100$, so $(n-2)(180^\circ) = (100-2)(180^\circ) = 98(180^\circ) = 17,640^\circ$.

Perimeter and Area

The last topic in plane geometry we consider is the **perimeters** and areas of figures in a plane.

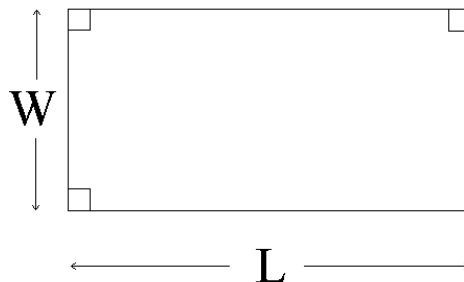
The **perimeter** is the total linear distance around the boundary of a polygon.

Example: the perimeter of the pentagon shown below is computed as $P = 3 \text{ in} + 6 \text{ in} + 5 \text{ in} + 3 \text{ in} + 5 \text{ in} = 22 \text{ in}$.



Area and Perimeter of a Rectangle

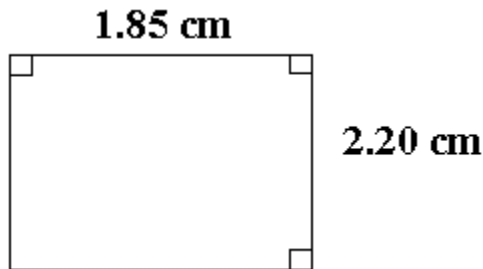
Consider a rectangle of width W and length L . The perimeter is computed as $P = W + W + L + L = 2W + 2L$.



The area of a rectangle, which measures the amount of “two dimensional space” inside the rectangle is given by $A = W \cdot L$.

Note : perimeter always has units of length, while area always has units of length².

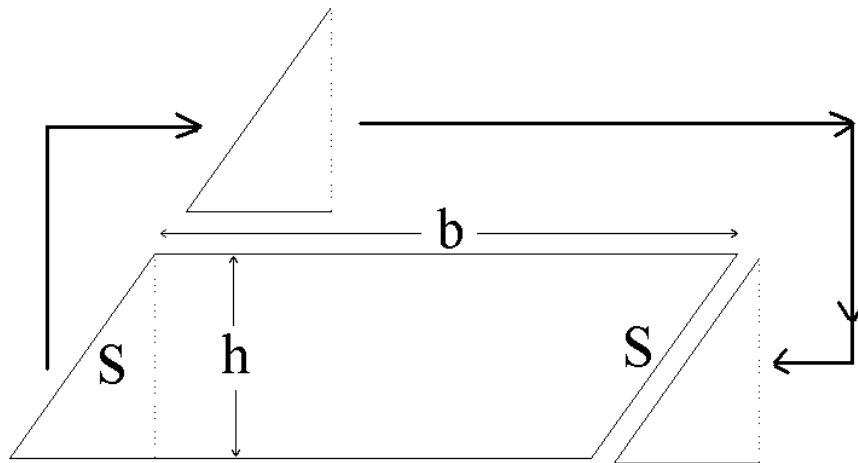
For the following rectangle, we compute P as $P = 2(1.85 \text{ cm}) + 2(2.20 \text{ cm}) = 8.10 \text{ cm}$.



The area is given by $A = 1.85 \text{ cm} \times 2.20 \text{ cm} = 4.07 \text{ cm}^2$.

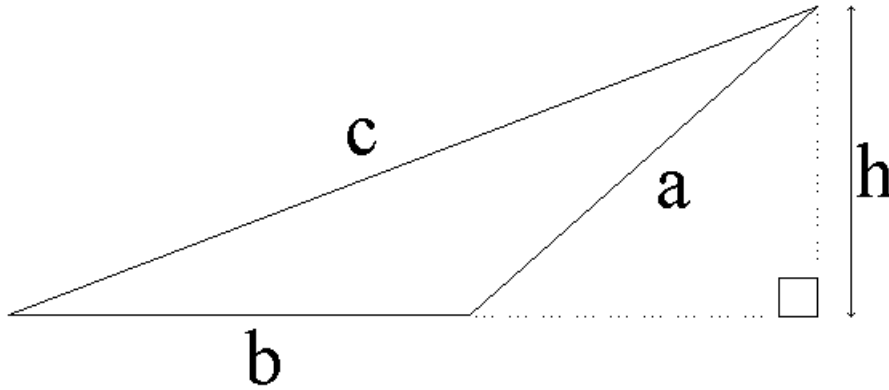
Area and Perimeter of a Parallelogram

A **parallelogram** is a four sided polygon (or quadrilateral) with opposite sides parallel. All rectangles are parallelograms, but in a generic parallelogram the angle between adjacent sides is not necessarily 90° . Consider the parallelogram shown below with base \mathbf{b} and perpendicular distance \mathbf{h} between the top and bottom sides. Imagine cutting off a right triangle from the left end and moving it to the right end. Since the left and right sides are parallel, this right triangle fits perfectly to make a rectangle of dimensions \mathbf{b} and \mathbf{h} . So we have for a parallelogram that $A = b \cdot h$.

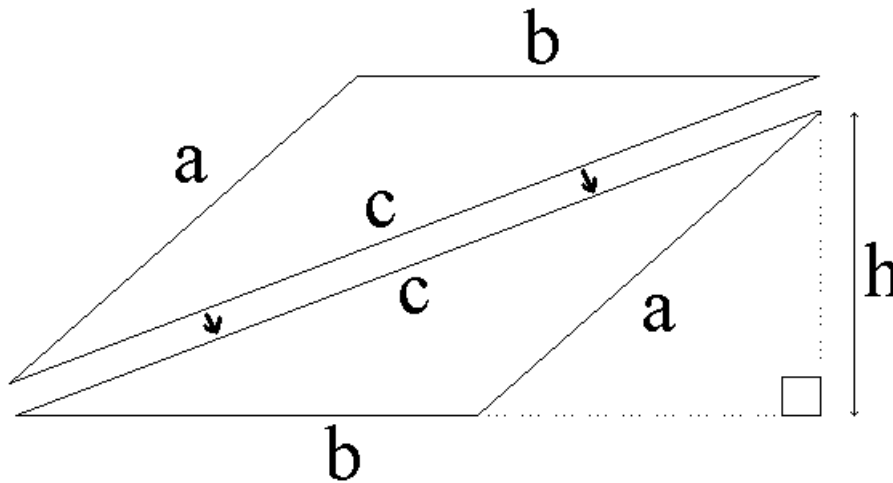


Area and Perimeter of a Triangle

Next consider a generic triangle with bottom side (base) of length **b** and perpendicular distance (height) **h** from the base to the top vertex.



Imagine making an exact copy of this triangle and joining it to the original triangle as shown.



The result is a parallelogram of base **b** and height **h**. Since the area of the original triangle is half of the area of this parallelogram, we arrive at the result that the area of a triangle is given by

$$A = \frac{1}{2}b \cdot h .$$

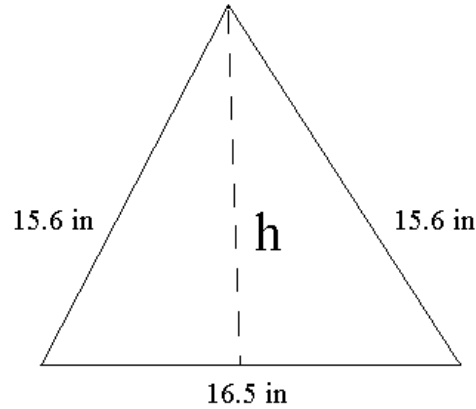
A more detailed argument shows that for a triangle with sides **a**, **b**, and **c** the area can be calculated from Heron's formula:

$$A = \sqrt{S(S-a)(S-b)(S-c)}$$

with the semi-perimeter **S** given by the formula: $S = \frac{a+b+c}{2}$.

EXAMPLE:

Consider calculating the perimeter and area of the following triangle.



The perimeter is just the sum of the lengths of the three sides.

$$P = 15.6 \text{ in} + 15.6 \text{ in} + 16.5 \text{ in} = 47.7 \text{ in}$$

To calculate the area, we could calculate the height by dropping a perpendicular from the top vertex. Since the triangle is isosceles, this bisects the 16.5 in base. Using the Pythagorean Theorem we compute h as follows:

$$h = \sqrt{(15.6 \text{ in})^2 - (16.5 \text{ in} \div 2)^2} = \sqrt{175.30 \text{ in}^2} = 13.2 \text{ in}$$

Then the area is calculated as half the base times the height.

$$A = 0.5 \times 16.5 \text{ in} \times 13.2 \text{ in} = 109 \text{ in}^2$$

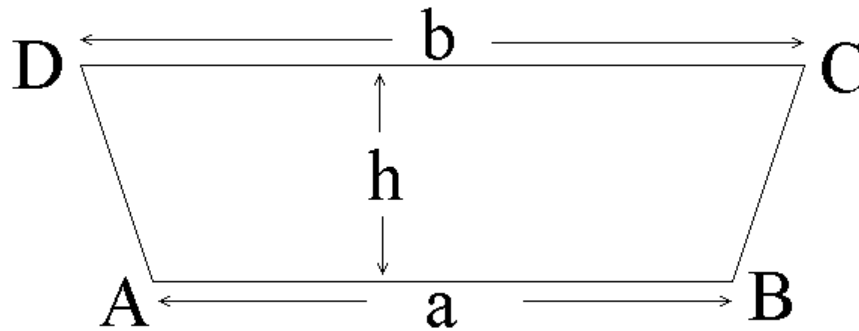
Another way to calculate the area is to use Heron's formula.

$$S = \frac{47.7 \text{ in}}{2} = 23.85 \text{ in}$$

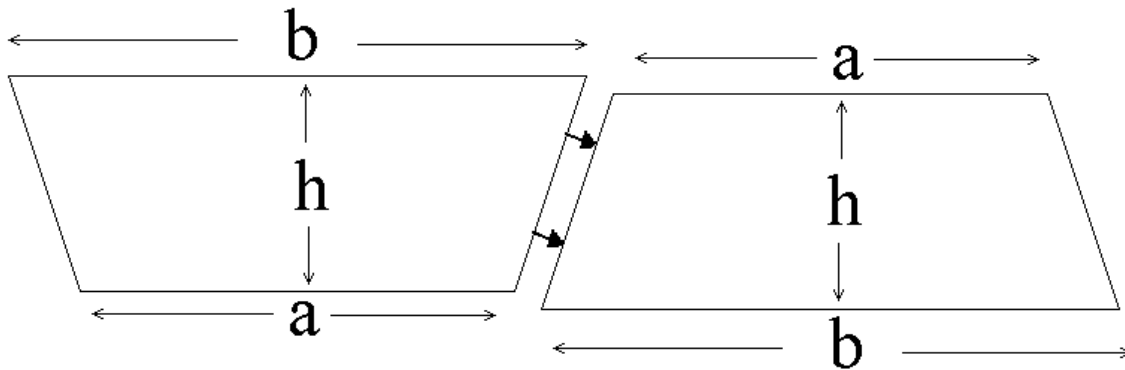
$$A = \sqrt{23.85 \text{ in}(23.85 \text{ in} - 15.6 \text{ in})(23.85 \text{ in} - 15.6 \text{ in})(23.85 \text{ in} - 16.5 \text{ in})} = \sqrt{11931 \text{ in}^4} = 109 \text{ in}^2$$

Area and Perimeter of a Trapezoid

A quadrilateral with two opposite sides parallel is called a trapezoid. Suppose that the two parallel faces have lengths \mathbf{a} and \mathbf{b} and are separated by a perpendicular distance \mathbf{h} .



Imagine making an exact copy of this trapezoid and joining it to the original trapezoid as shown.

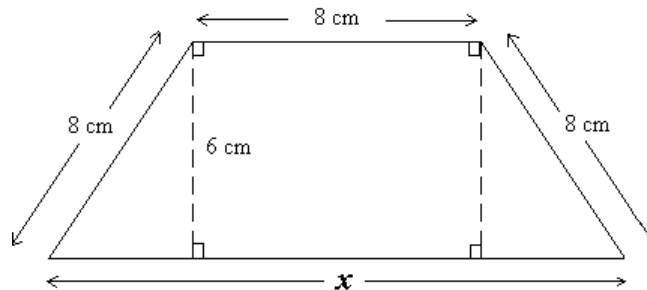


The result is a parallelogram of base $a + b$ and height h . Since the area of the original trapezoid is half of the area of this parallelogram, we arrive at the result that the area of a trapezoid is given by the following formula:

$$A = \frac{1}{2}(a + b) \cdot h$$

EXAMPLE:

As an application consider calculating the perimeter and area of the following trapezoid.



The first step is to calculate the length x . Using the Pythagorean Theorem the length of the base of the right triangles that form the sides of the trapezoid is computed as follows:

$$\sqrt{(8 \text{ cm})^2 - (6 \text{ cm})^2} = \sqrt{28 \text{ cm}^2} = 5.29 \text{ cm} .$$

Then $x = 8 \text{ cm} + 2 \times 5.29 \text{ cm} = 18.58 \text{ cm}$. The perimeter is just the sum of the lengths of the four sides. $P = 18.6 \text{ cm} + 8 \text{ cm} + 8 \text{ cm} + 8 \text{ cm} = 42.6 \text{ cm}$.

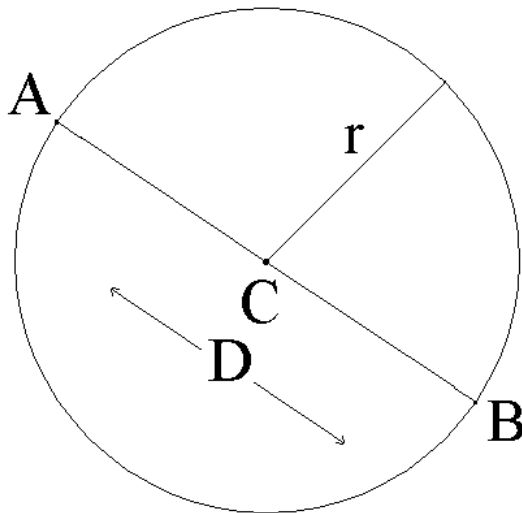
To calculate the area, we could add the area of the two right triangles that form the sides of the trapezoid to the area of the 8 cm by 6 cm central rectangle.

$$A = 2 \times \frac{1}{2} \times 5.29 \text{ cm} \times 6 \text{ cm} + 8 \text{ cm} \times 6 \text{ cm} = 79.7 \text{ cm}^2$$

We get the same result by using the formula for the area of a trapezoid .

$$A = \frac{1}{2} \times (18.58 \text{ cm} + 8 \text{ cm}) \times 6 \text{ cm} = 79.7 \text{ cm}^2$$

Circles



$$D = 2 \cdot r$$

$$r = \frac{D}{2}$$

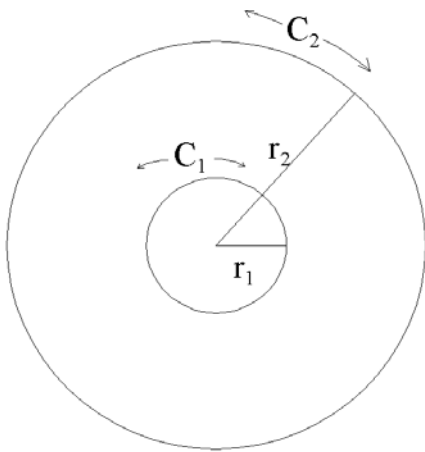
A circle is formed by generating all points in a plane which are a fixed distance called the **radius** from a center point here labeled as **C**. A line segment with end points on the circle that passes through the center is called a **diameter**. **D** and **r** symbolize the lengths of the diameter and radius, respectively.

Since $AC = CB = r$ and $D = AB = AC + CB = 2r$, we have the following formulas:

$$D = 2r \text{ and } r = \frac{D}{2} .$$

All circles are **similar**. By this we mean that all circles have the same shape. The distance around the boundary of a circle is called its **circumference** which is like the perimeter of a polygon.

Imagine that we have two circles. The first labeled as **1** has radius r_1 , diameter D_1 and circumference C_1 . The second labeled as **2** has radius r_2 , diameter D_2 and circumference C_2 . Since all circles are scale models of each, the ratio of circumference to diameter is the same for both circles. The value of this ratio is symbolized by the Greek letter lower case pi. Pi is symbolized by $\pi \approx 3.141592654$. The value of pi is slightly larger than 3. In fact the decimal approximation to pi is known to over a billion digits! There may be a pi button on your calculator. Pi is approximately equal to 3.14159. However 3.14 will be good enough in most cases.



$$\frac{C_2}{D_2} = \frac{C_1}{D_1} = \pi \quad \text{or} \quad \begin{aligned} C_2 &= \pi \cdot D_2 \\ C_1 &= \pi \cdot D_1 \end{aligned}$$

According to the above argument then, the distance around the outside of a circle or in other words the “perimeter” of a circle, is given by the formula.

$$C = \pi \cdot D = 2\pi \cdot r$$

The “perimeter” of a circle is technically called the **circumference** of the circle.

Area of a Circle

$$A = r \cdot \frac{C}{2} = r \cdot \frac{2\pi r}{2} = r \pi r = \pi r^2$$

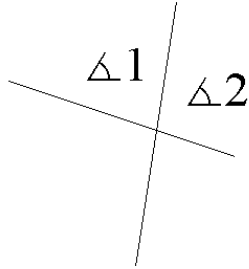
EXAMPLE:

The circumference and area of a circle of diameter 2.500 in are computed by the following calculations:

$$\begin{aligned} C &= \pi \times 2.500 \text{ in} = 7.854 \text{ in} \\ r &= \frac{D}{2} = \frac{2.500 \text{ in}}{2} = 1.250 \text{ in} \\ A &= \pi \times (1.250 \text{ in})^2 = 4.909 \text{ in}^2 \end{aligned}$$

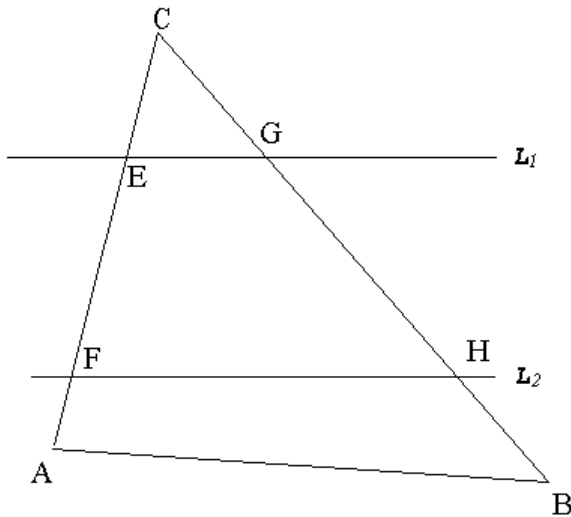
Your Turn!!

If $\angle 1 = 85^\circ 19' 56''$, what is the measure of $\angle 2$?



1. $\angle 2 =$ _____

If line L_1 is parallel to line L_2 and $\angle CAB = 76^\circ 59'$ and $\angle CBA = 46^\circ 29'$ and $\angle CEG = 61^\circ 29'$, find the requested missing \angle 's.

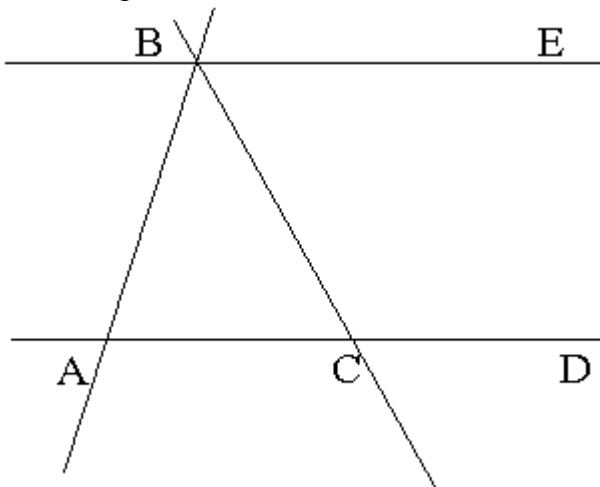


2. $\angle ACB =$ _____

3. $\angle CFH =$ _____

4. $\angle CGE =$ _____

If BE is parallel to AD and $\angle ACB = 64^\circ 17'$ and $\angle BAC = 66^\circ 35'$, find the missing \angle 's.

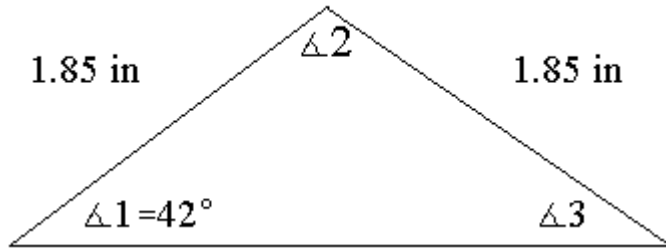


5. $\angle ABC =$ _____

6. $\angle BCD =$ _____

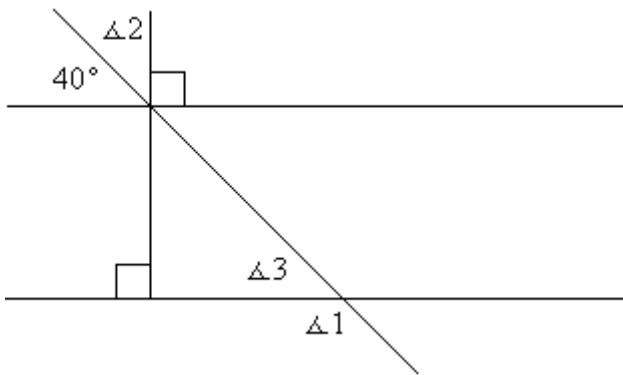
7. $\angle CBE =$ _____

Find the measure of the missing angles.



8. $\angle 2 =$ _____

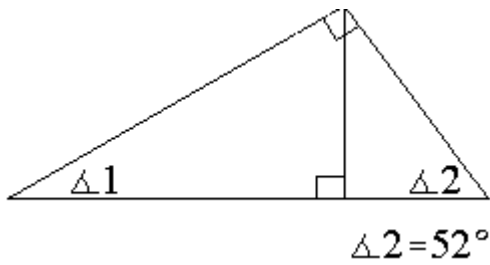
9. $\angle 3 =$ _____



10. $\angle 1 =$ _____

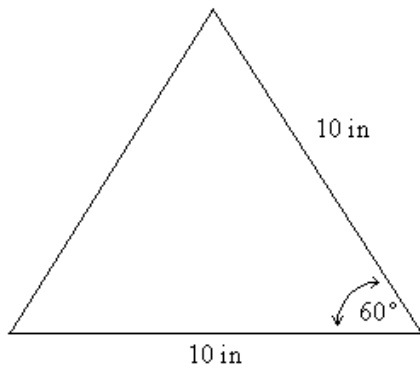
11. $\angle 2 =$ _____

12. $\angle 3 =$ _____

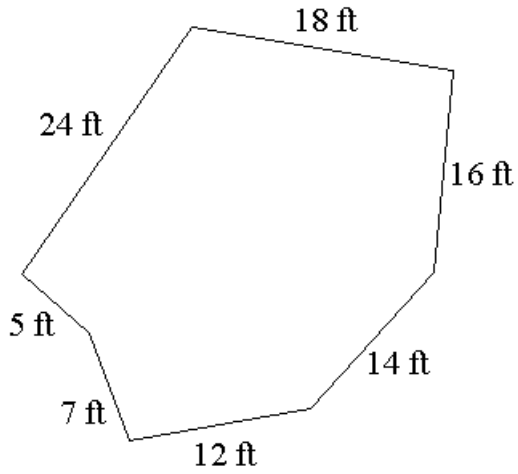


13. $\angle 1 =$ _____

For each figure below calculate the distance around (perimeter) P.

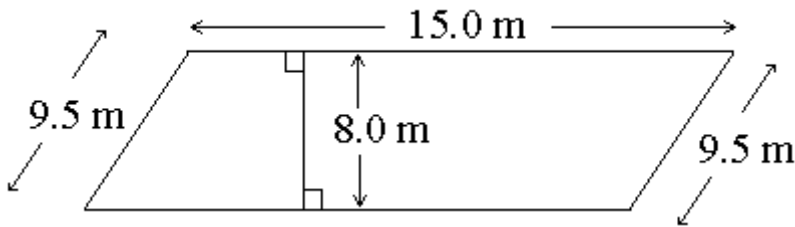


14. $P =$ _____



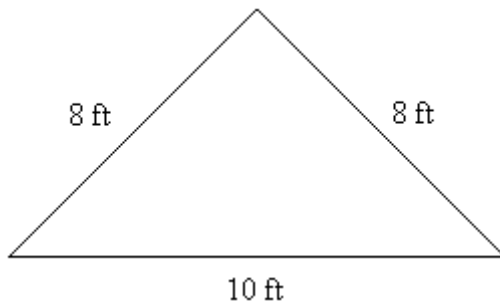
15. $P =$ _____

For each figure below calculate both the area, A , and the distance around (perimeter or circumference), P or C .



16. $P =$ _____

17. $A =$ _____



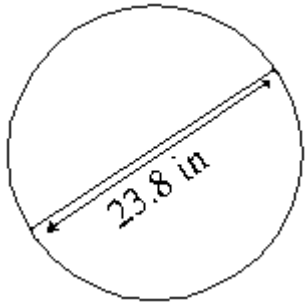
18. $P =$ _____

19. $A =$ _____



20. $P =$ _____

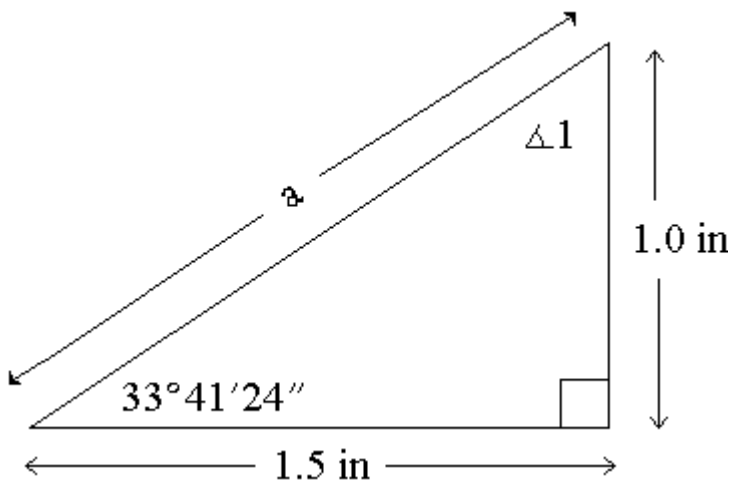
21. $A =$ _____



22. $C =$ _____

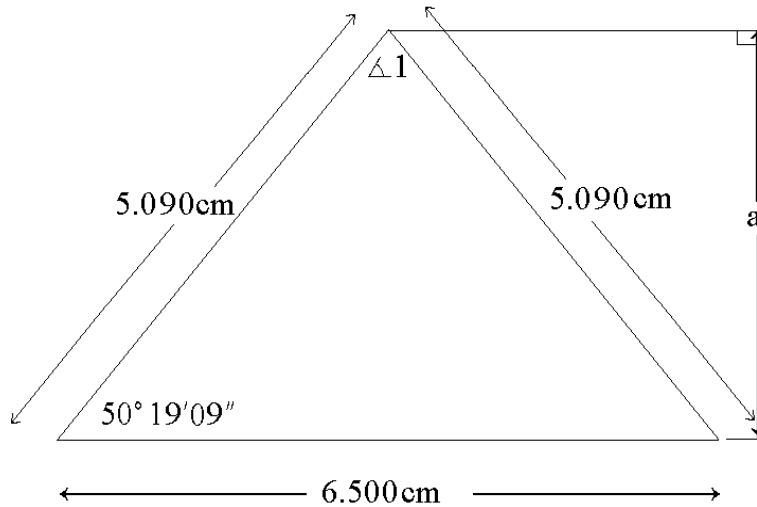
23. $A =$ _____

For each figure below calculate the requested missing information.



24. $\angle 1 =$ _____

25. $a =$ _____



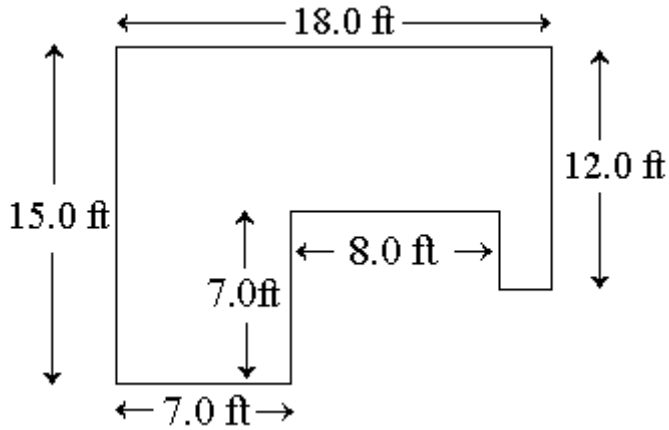
26. $\angle 1 =$ _____

27. $a =$ _____

28. $P =$ _____

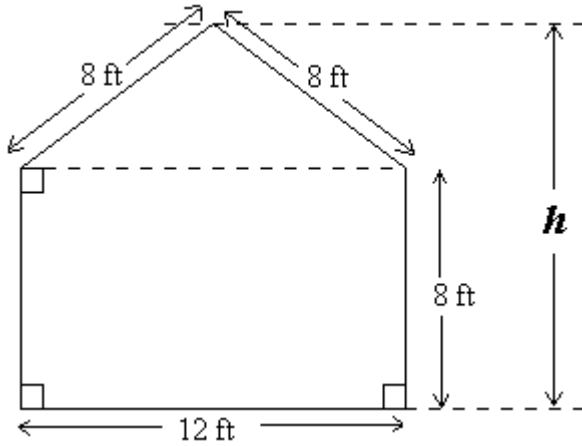
29. $A =$ _____

How many square feet of flooring does the following room have?



30. _____

The cross section of a shed is shown below. Determine the height h above the ground and A , the total area of the building's cross section.



31. $h =$ _____

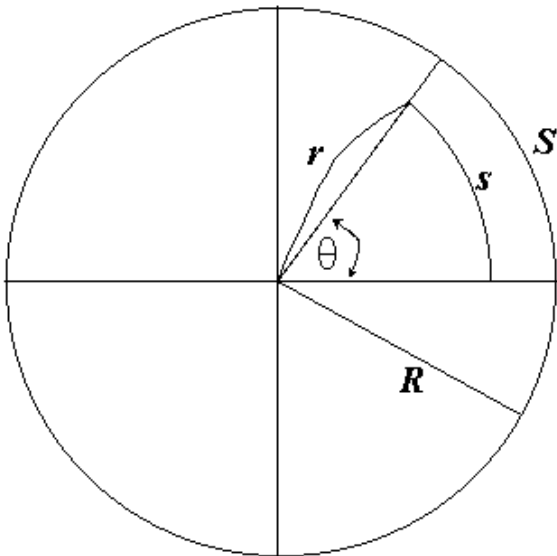
32. $A =$ _____

A 2.25 cm diameter hole is drilled in a 4.5 cm diameter circle. What area is left after the hole has been drilled?

33. _____

6.2 - Radian Measure and its Applications

In addition to DMS (Degrees, Minutes, and Seconds) and DD (Decimal Degrees), there is another system for measuring angles called **radian measure**. The basis of this system is that all sectors of a circle having the same central angle θ (theta) are similar. Thus, for a given central angle, the ratio of the arc length of the sector to the radius of the circle is a constant. We define this constant as the radian measure of the angle. We say that the arc length is “cut-off” or subtended by the central angle theta at a given radius r .



$$\theta(\text{measured in radians}) = \frac{s}{r} = \frac{S}{R}$$

$$\text{Arc length} = \text{radius} \times \text{central angle in radians}$$

$$s = r \cdot \theta \quad (\theta \text{ measured in radians})$$

$$C = 2\pi r = r \cdot (\text{full revolution angle in radians})$$

so a full revolution of $360^\circ = 2\pi$ radians
or more simply π rads = 180°

To convert decimal degree measurements to radians we use the conversion factor $1 = \frac{\pi \text{ radians}}{180^\circ}$.

To convert radian measure to decimal degrees we use the conversion factor $1 = \frac{180^\circ}{\pi \text{ radians}}$.

The following examples illustrate the use of these conversion factors.

$$34^\circ 17' 45'' = \left(34 + \frac{17}{60} + \frac{45}{3600} \right)^\circ = 34.2958^\circ = 34.2958^\circ \cdot \frac{\pi \text{ radians}}{180^\circ} = 0.5986 \text{ rads}$$

$$\frac{\pi}{8} \text{ radians} = \frac{\pi \text{ rads}}{8} \cdot \frac{180^\circ}{\pi \text{ rads}} = \frac{180^\circ}{8} = 22.5^\circ = 22^\circ 30' 00'' = 22.5^\circ \cdot \frac{1 \text{ rev}}{360^\circ} = 0.0625 \text{ rev}$$

$$0.912 \text{ rads} = 0.912 \text{ rads} \cdot \frac{180^\circ}{\pi \text{ rads}} = 52.25375^\circ = 52^\circ 15' 13'' = 52.25375^\circ \cdot \frac{1 \text{ rev}}{360^\circ} = 0.1451 \text{ rev}$$

The next three examples use these principles.

Example: A car on a circular track of radius 0.4 miles is travelling 125 mph. In 2 seconds through what central angle measured from the center of the track does the car travel?

Solution: The arc length subtended by the central angle is just the distance travelled by the car. Since distance is speed multiplied by time, we have

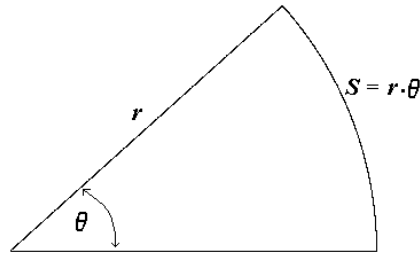
$$S = \left(\frac{125 \text{ miles}}{1 \text{ hr}} \cdot \frac{1 \text{ hr}}{60 \text{ min}} \cdot \frac{1 \text{ min}}{60 \text{ sec}} \right) \cdot 2 \text{ sec} = .0694 \text{ miles}$$

$$\theta (\text{in radians}) = \frac{S}{r}, \text{ so}$$

$$\theta (\text{in radians}) = \frac{0.0694 \text{ miles}}{0.4 \text{ miles}} = 0.174 \text{ rads} = 0.174 \text{ rads} \cdot \frac{180^\circ}{\pi \text{ rads}} = 9.97^\circ \approx 10^\circ$$

Example: What is the area of the sector of a circle of diameter 12.0 m subtended by an angle of 42 degrees?

Solution: A sector of a circle is “wedge” with a vertex at the circle’s center as shown below.



The fraction of a full circle that this sector represents is $\frac{42^\circ}{360^\circ} = 0.1167$. Thus, the area of the sector will be 11.67% of the area of a full circle of radius 6.0 m, so

$$A = 0.1167 \cdot \pi \cdot (6.0 \text{ m})^2 = 13.2 \text{ m}^2.$$

Example: If a 34.0 in diameter tire on a car makes 400 rpm and never slips, how fast is the car moving?

Solution: If the tire never slips the distance the tire travels along the ground in one revolution is equal to the circumference of the tire. If the wheel makes 400 revolutions in one minute, then it is moving at a speed of 400 circumferences per minute.

$$\begin{aligned} \text{Speed} &= \frac{400 \text{ rev}}{1 \text{ min}} \cdot \frac{\pi \cdot 34.0 \text{ in}}{1 \text{ rev}} = 42,700 \frac{\text{in}}{\text{min}} = 42,700 \frac{\text{in}}{\text{min}} \cdot \frac{1 \text{ ft}}{12 \text{ in}} = 3560 \frac{\text{ft}}{\text{min}} \\ &= 3560 \frac{\text{ft}}{\text{min}} \cdot \frac{60 \text{ min}}{1 \text{ hr}} \cdot \frac{1 \text{ mile}}{5280 \text{ ft}} = 40.6 \text{ mph} \end{aligned}$$

Your Turn!!

1. Convert the following angles from degree measure to radians and revolutions. Give answers to four decimal places.

	Radians	Revolution
30°	= _____	_____
120°	= _____	_____
300°	= _____	_____

2. Convert the following angles from radian measure to degree measure and revolutions. Give answers to four decimal places.

	Dec Degrees	DMS	Revolution
$\frac{\pi}{2}$	= _____	_____	_____
$\frac{\pi}{12}$	= _____	_____	_____
0.50	= _____	_____	_____

3. To three digits find the arclength subtended by an angle of 43.5° and a radius of 31.7 in.

4. Find the angle in both radians and degrees subtended by an arc of 40.0 cm in a circle of diameter 50.0 cm .

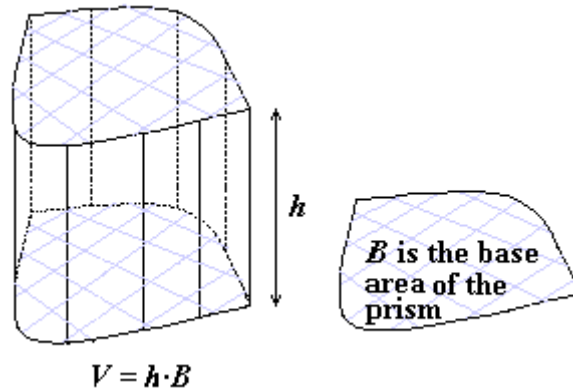
5. A car on a circular race track with a radius of 0.25 miles is travelling 110 mph. In six seconds through what central angle from the center of the track does the car turn?

6. Find the area of a sector of a circle of diameter 40.0 ft subtended by an angle of 16 degrees.

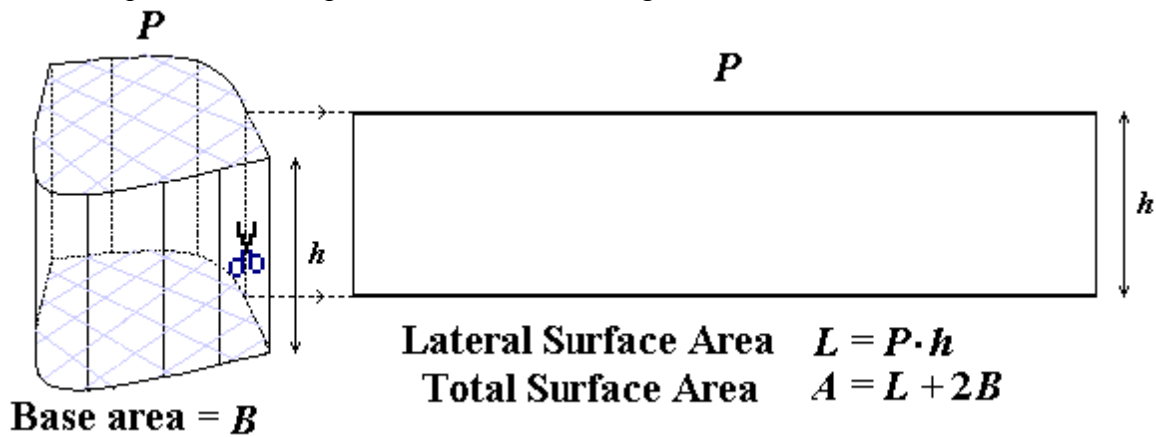
7. A bicyclist pedals at such a rate that both wheels rotate at 205 rpm. The outside wheel diameter is 26.0 in . Assuming that the tires never slip against the ground, what is the bicyclist's speed in mph ?

6.3 -The Volume and Surface Area of a Solid

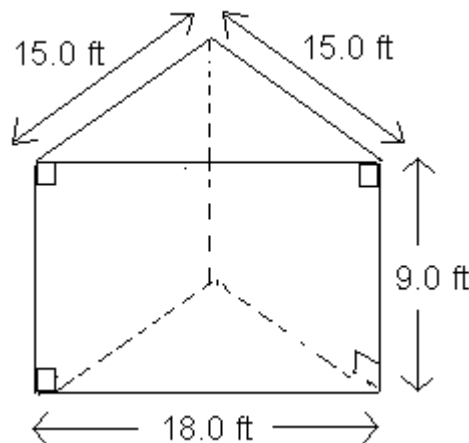
The core idea of a **volume** is an area moved through a third dimension. Consider the following prism. This is a solid figure with two parallel faces and lateral (sideways) sides perpendicular to these faces. The parallel face is called the prism's base. If the area of the base is B and the perpendicular height between the two parallel faces is h , then the volume of the prism is given by the formula $V = h \cdot B$. Volume as the product of a length times an area **must have units of cubic length**, such as cubic inches, cubic feet etc. . . .



The area occupied by the lateral sides of the prism is called the **lateral surface area**, L . The sum of the lateral surface area with the top and bottom base areas is called the **total surface area**, A . If you imagine “cutting” along the side of the prism perpendicular to the base and then “unfolding” and laying flat the lateral surface area, the resulting figure is a rectangle with dimensions equal to the base perimeter, P , and the height, h .



Example: consider calculating the volume, lateral and total surface areas of the following prism.



The base is a triangle of sides 15.0 ft, 15.0 ft and 18.0 ft. The base area can be calculated using Heron's formula.

$$S = \frac{15.0 + 15.0 + 18.0}{2} \text{ ft} = 24.0 \text{ ft}$$

$$B = \sqrt{24.0 \times (24.0 - 15.0) \times (24.0 - 15.0) \times (24.0 - 18.0)} \text{ ft}^2$$

$$B = 108 \text{ ft}^2$$

$$V = 108 \text{ ft}^2 \times 9.0 \text{ ft} = 972 \text{ ft}^3$$

$$L = 48.0 \text{ ft} \times 9.0 \text{ ft} = 432 \text{ ft}^2$$

$$A = 432 \text{ ft}^2 + 2 \times 108 \text{ ft}^2 = 648 \text{ ft}^2$$

An alternate approach to calculating this base area uses the fact that the base is an isosceles triangle. Dropping a perpendicular from the top vertex will then bisect the 18.0 ft side. The height of the resulting right triangle is then calculated by use of the Pythagorean Theorem.

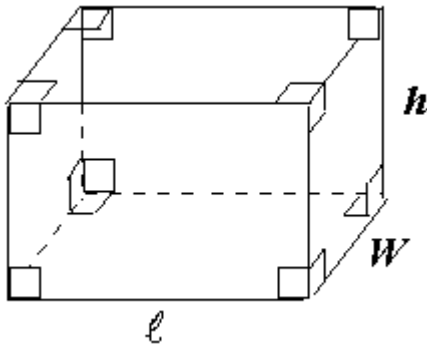
$$\sqrt{15^2 - 9^2} \text{ ft} = \sqrt{144} \text{ ft} = 12.0 \text{ ft}$$

$$B = \frac{1}{2} 18.0 \text{ ft} \times 12.0 \text{ ft} = 108 \text{ ft}^2$$

The formulas for the volumes and surface areas of solids, which are not prisms, require more elaborate geometric reasoning. These are included in the results presented below.

Volume and Surface Area of a Rectangular Prism

Probably the most recognizable prism is the rectangular prism or "box". The volume is simply the product of the lengths of the three sides.



$$B = \ell \cdot W$$

$$P = 2\ell + 2W$$

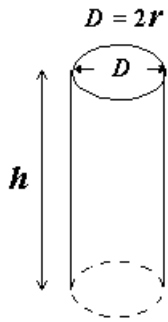
$$V = B \cdot h = \ell \cdot W \cdot h$$

$$L = P \cdot h = 2\ell \cdot h + 2W \cdot h$$

$$A = L + 2B = 2\ell \cdot h + 2W \cdot h + 2\ell \cdot W$$

Volume and Surface Area of a Cylinder

A right circular cylinder is also a prism. Here the circumference of the base circle is used in calculating the lateral surface area.



$$B = \pi r^2 = \frac{\pi D^2}{4}$$

$$C = \pi D = 2\pi r$$

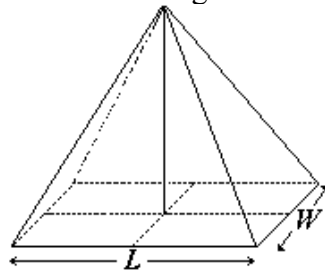
$$V = B \cdot h = \pi r^2 h = \frac{\pi D^2 h}{4}$$

$$L = C \cdot h = \pi D h = 2\pi r h$$

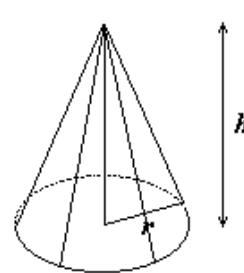
$$A = L + 2B = 2\pi r h + 2\pi r^2 = \pi D h + \frac{\pi D^2}{2}$$

Volume of a Pyramid and Cone

Right **pyramids** and **cones** both have a volume equal to *one third* that of the corresponding prism having the same base and height.

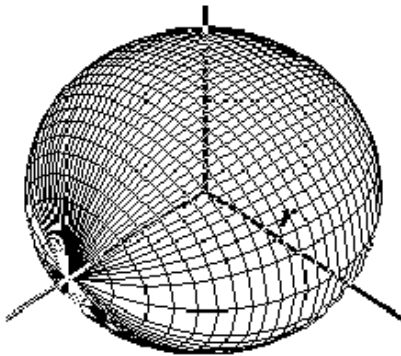


$$V = \frac{h \cdot B}{3} = \frac{h \cdot L \cdot W}{3}$$



$$V = \frac{h \cdot B}{3} = \frac{\pi r^2 h}{3}$$

Volume and Total Surface Area, A, of a Sphere



$$V = \frac{4\pi r^3}{3} = \frac{\pi D^3}{6}$$

$$A = 4\pi r^2 = \pi D^2$$

Example: A storage tank is in the shape of a right circular cylinder capped with a hemispherical dome. The inner diameter of the tank is 15.0 ft and the height from the floor to the inner top of the hemisphere is 25.0 ft. The entire inside area of the tank needs to be coated with a sealant. One gallon of this sealant is required for every 150 square ft. What is the capacity of the tank in gallons? How many gallons of sealant are required to cover the inside surface of the tank?

Solution: The radius of the hemisphere must match the radius of the cylinder, $r = 7.5$ ft. The height of the cylinder is the total height, 25 ft, minus this radius, which amounts to 17.5 ft. The capacity or volume of the tank is the sum of the cylinder's volume plus half of the volume of a sphere with a 7.5 ft radius. (To convert cubic ft to gallons see Unit 5 on Measurement.)

$$\begin{aligned} V &= \pi r^2 h + \frac{1}{2} \cdot \frac{4\pi r^3}{3} = \pi (7.5 \text{ ft})^2 \cdot 17.5 \text{ ft} + \frac{2\pi (7.5 \text{ ft})^3}{3} \\ &= 3976 \text{ ft}^3 = 3976 \text{ ft}^3 \cdot \frac{1 \text{ gal}}{0.13368 \text{ ft}^3} = 29,743 \text{ gal} \end{aligned}$$

Given that the tank's dimensions are only good to three digits, we should probably report the tank's capacity as 29,700 gallons.

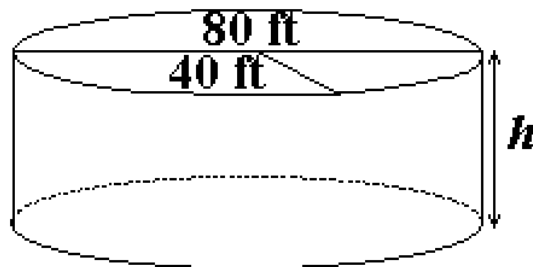
To determine how many gallons of sealant are required we need to calculate the total internal surface area of the tank. This consists of the circular floor, the lateral surface of the cylinder and half the surface area of a sphere with a 7.5 ft radius.

$$A = \pi r^2 + 2\pi r \cdot h + \frac{1}{2} \cdot 4\pi r^2 = \pi (7.5 \text{ ft})^2 + 2\pi \cdot 7.5 \text{ ft} \cdot 17.5 \text{ ft} + 2\pi (7.5 \text{ ft})^2 = 1355 \text{ ft}^2$$

One gallon of sealant covers 150 square feet of surface, so we will require

$1355 \text{ ft}^2 \times \frac{1 \text{ gal}}{150 \text{ ft}^2} = 9.03 \text{ gal}$. However, to be safe and to cover waste 10 gallons should probably be ordered.

Example: As a variation on the above problem, suppose we were designing a cylindrical storage tank (no hemispherical cap this time!) to have a capacity of 1 million gallons and a base diameter of 80 ft, how tall must the tank be?



Solution: First, we convert the million gallons into cubic feet.

$$1,000,000 \text{ gal} = 10^6 \text{ gal} \cdot \frac{0.13368 \text{ ft}^3}{1 \text{ gal}} = 133,680 \text{ ft}^3 = 134,000 \text{ ft}^3$$

Then from the formula for the volume we solve for the height h .

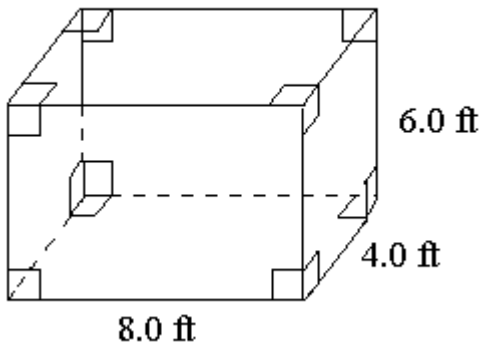
$$V = B \cdot h = \pi r^2 \cdot h$$

$$\frac{V}{\pi r^2} = \frac{\pi r^2 \cdot h}{\pi r^2} = h \quad \text{or} \quad h = \frac{V}{\pi r^2} = \frac{134000 \text{ ft}^3}{\pi (40 \text{ ft})^2} = 26.7 \text{ ft}$$

Your Turn!!

Find the lateral surface area, L , the total surface area, A , and the the volume, V , of the following solids.

1.

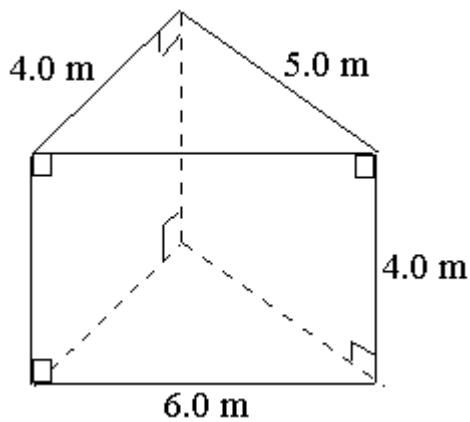


$L =$ _____

$A =$ _____

$V =$ _____

2.

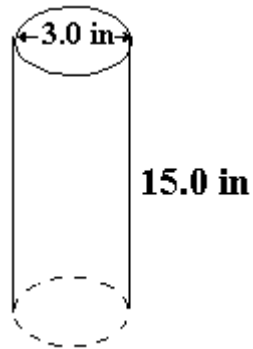


$L =$ _____

$A =$ _____

$V =$ _____

3.

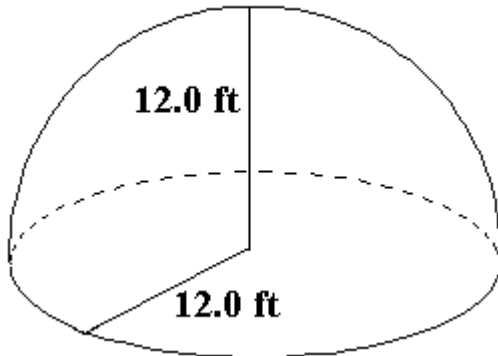


$$L = \underline{\hspace{2cm}}$$

$$A = \underline{\hspace{2cm}}$$

$$V = \underline{\hspace{2cm}}$$

4.



$$L = \underline{\hspace{2cm}}$$

$$A = \underline{\hspace{2cm}}$$

$$V = \underline{\hspace{2cm}}$$

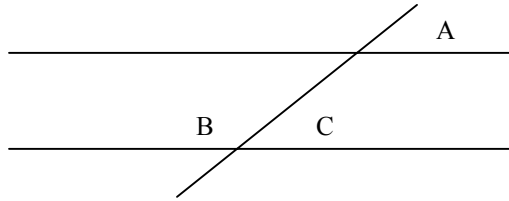
A cylindrical holding tank has an inner diameter of 40.0 ft and walls that are 18 in thick. The tank is designed to hold 350,000 gallons.

5. What is the tank's inside perimeter?
6. What is the tank's outside perimeter?
7. What is the area of the tank floor?
8. What is the height of the tank in feet?

9. A swimming pool is in the shape of a trapezoidal prism. The shallow end is 3.0 ft deep, the deep end is 8.0 ft, the width of the pool is 25 ft and the length from the shallow end to the deep end is 50 ft. How many gallons of water does it take to fill the pool?

Chapter 6 Practice Exam

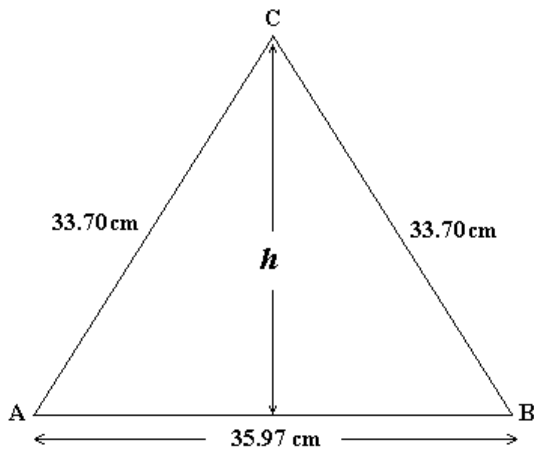
1. Given the figure below and the fact that angle $A = 31^\circ$, Find angles B and C



B = _____

C = _____

2. Find the Perimeter, height and area of the following triangle. State four digits in your answers for the height and area.

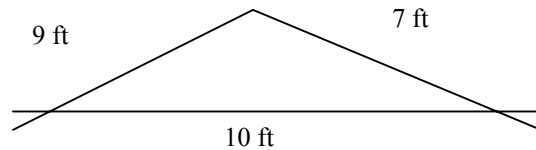


a. Perimeter _____

b. Height, h _____

c. Area _____

3. Find the Area and Perimeter of the triangle below. Give the area to four digits.



a. Area _____ (Hint: use Heron's Formula)

b. Perimeter _____

4. For a circle with a diameter of 12 inches, find the following

- a. Area _____ (Round answer to one decimal place)
- b. Circumference _____ (Round answer to one decimal place)
- c. The Arclength for of a sector of the circle mentioned above subtended by an angle of 10° (Remember that the angle must be in radians)
_____ (Round answer to one decimal place)
- d. Find the area of a sector formed by an angle of 10° for the circle above(Remember that the angle must be in radians)
_____ (Round answer to one decimal place)

5. Convert 1.5300 radians to decimal degrees, DMS, and Revolutions

- a. Decimal Degrees _____ (Round answer to four decimal places)
- b. DMS(Degrees, Minutes, Seconds) _____
- c. Revolutions _____ (Round answer to four decimal places)

6. A right circular cylinder with base of radius 15 feet and a height of 3 feet, find:

- a. Volume _____ (Round answer to one decimal place)
- b. Total Surface Area _____ (Round answer to one decimal place)

7. Find the volume of a cone with a base of diameter 5 feet and a height of 3 feet

Volume _____ (Round answer to one decimal place)

8. Find the volume and total surface area of a cube with sides of length 11 feet

a. Volume _____ (Round answer to one decimal place)

b. Total Surface Area _____ (Round answer to one decimal place)

9. A sphere that you could just fit inside the cube in the previous problem, find:

a. Volume _____ (Round answer to one decimal place)

b. Surface Area _____ (Round answer to one decimal place)

c. Find the cost of coving the sphere in paint that cost \$0.10 per square foot.
(Round answer to the nearest cent)

Chapter 6 Answer Section

Section 6.1 Solution Set:

1. $\angle 2 = 94^\circ 40' 04''$; 2. $\angle ACB = 56^\circ 32'$; 3. $\angle CFH = \angle CEG = 61^\circ 29'$
4. $\angle CGE = 180^\circ - \angle ACB - \angle CEG = 61^\circ 59'$
5. $\angle ABC = 180^\circ - \angle BAC - \angle ACB = 49^\circ 08'$
6. $\angle BCD = 180^\circ - \angle ACB = 115^\circ 43'$; 7. $\angle CBE = \angle ACB = 64^\circ 17'$
8. $\angle 2 = 180^\circ - 84^\circ = 96^\circ$; 9. $\angle 3 = 42^\circ$; 10. $\angle 1 = 180^\circ - 40^\circ = 140^\circ$; 11. $\angle 2 = 50^\circ$
12. $\angle 3 = 40^\circ$; 13. $\angle 1 = 38^\circ$; 14. $P = 30$ in; 15. $P = 96$ ft; 16. $P = 49.0$ m; 17. $A = 120$. m^2
18. $P = 26$ ft; 19. $A = 31.2$ ft^2 ; 20. $P = 58.5$ in; 21. $A = 152$. in^2 ; 22. $C = 74.8$ in
23. $A = 445$. in^2 ; 24. $\angle 1 = 56^\circ 18' 36''$; 25. $a = 1.80$ in; 26. $\angle 1 = 79^\circ 21' 42''$
27. $a = 3.92$ cm; 28. $P = 16.7$ cm; 29. $A = 12.7$ cm^2 ; 30. 205 ft^2 ; 31. $h = 13.3$ ft
32. $A = 128$. ft^2 ; 33. 11.9 cm^2

Section 6.2 Solution Set:

1.	Radians	Revolution	2.	Dec Degrees	DMS	Revolution
	$30^\circ = 0.5236$	0.08333	$\frac{\pi}{2}$	$= 90^\circ$	$90^\circ 0' 00''$	0.250
	$120^\circ = 2.0944$	0.3333	$\frac{\pi}{12}$	$= 15^\circ$	$15^\circ 0' 00''$	0.0417
	$300^\circ = 5.2360$	0.8333	0.50	$= 28.6479^\circ$	$28^\circ 38' 52''$	0.0796

3. 24.1 in;
4. 1.60 rads = 91.673 degrees;
5. 0.733 rads = 42.0 degrees;
6. 55.9 ft^2 ; 7. 15.9 mph

Section 6.3 Solution Set:

1. $L = 144$ ft^2 , $A = 208$ ft^2 , $V = 192$ ft^3 ;
2. $L = 60$ m^2 , $A = 79.8$ m^2 , $V = 39.7$ m^3
3. $L = 141.4$ in^2 , $A = 155.5$ in^2 , $V = 106.0$ in^3 ;
4. $L = 904.8$ ft^2 , $A = 1357.2$ ft^2 , $V = 3619.1$ ft^3
5. 125.7 ft; 6. 135.1 ft; 7. 1256 ft^2 ; 8. 37.2 ft; 9. 51,430 gallons

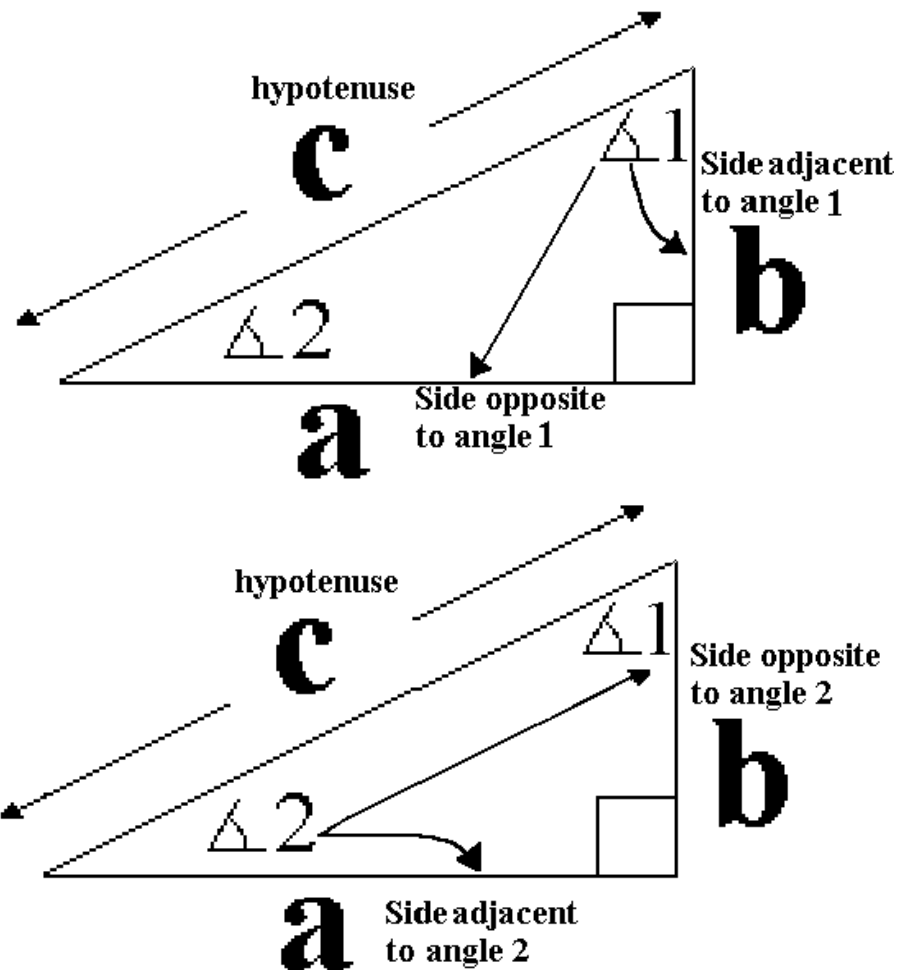
Chapter 6 Practice Exam Solution Set:

1. $B = 149^\circ$, $C = 31^\circ$
2. a. 103.37 cm b. 28.50 cm c. 512.6 cm²
3. a. 30.59 ft² b. 26 ft
4. a. 113.0 in² b. 37.68 in c. 1.0 in d. 3.14 in²
5. a. 87.6625° b. 87°39'45" c. 0.2435 revolutions
6. a. 2119.5 ft³ b. 1695.6 ft²
7. 19.6 ft³
8. a. 1331 ft³ b. 726 ft²
9. a. 696.6 ft³ b. 379.94 ft² c. \$37.99

Chapter 7 - Trigonometry

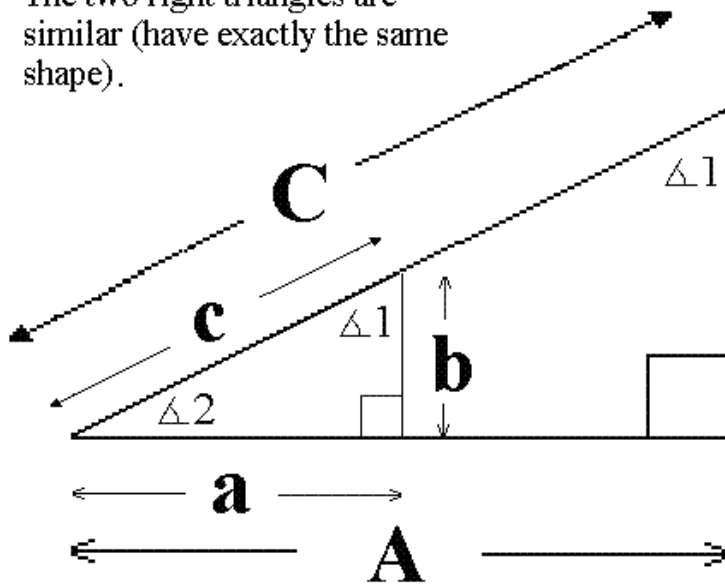
Section 7.1 Sine, Cosine and Tangent

In this section we will restrict our attention to right triangles. With respect to the acute angles (the two angles less than 90 degrees) in a right triangle, we can classify the two legs (the sides which are not the largest side, the hypotenuse) as being either **opposite** to or **adjacent** to the given angle. This is shown below in the following two diagrams.



Since angles 1 and 2 are complementary, the value of one angle determines the second. Two triangles with the same set of angles are similar. They must have exactly the same shape. This shape then depends entirely on the value of one of the two acute angles. We say the shape is a “function” of the acute angle. The angle determines the shape and hence the various proportions between the triangle’s three sides. Thus these proportions are functions of the acute angle. We call these functions trigonometric functions (trig functions for short) and for a given acute angle there are three of special importance. These are the “sine”, “cosine” and “tangent” functions.

The two right triangles are similar (have exactly the same shape).



Ratios of corresponding sides are equal.

$$\begin{array}{c} \uparrow \\ \mathbf{B} \\ \downarrow \end{array} \quad \frac{\mathbf{b}}{\mathbf{a}} = \frac{\mathbf{B}}{\mathbf{A}} \quad \frac{\mathbf{b}}{\mathbf{c}} = \frac{\mathbf{B}}{\mathbf{C}} \quad \frac{\mathbf{a}}{\mathbf{c}} = \frac{\mathbf{A}}{\mathbf{C}}$$

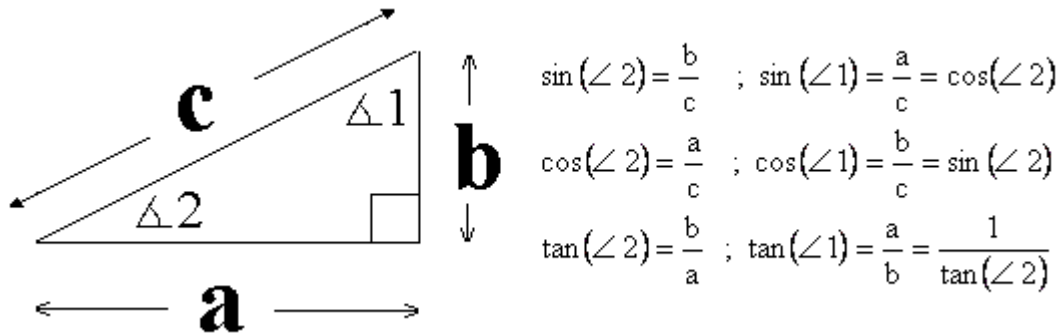
The common abbreviation for sine is “sin” (still pronounced sine). The cosine means the sine of the complement (see the picture above) and is abbreviated “cos”, while tangent is written as “tan”. Standard notation is to separate the name of the trig function from the name of the acute angle (called the “argument” of the function) with parenthesis. The idea is that, if you “input” the value of the acute angle to the trig function, it “outputs” the requested ratio of sides.

Definition of the Trigonometric Functions

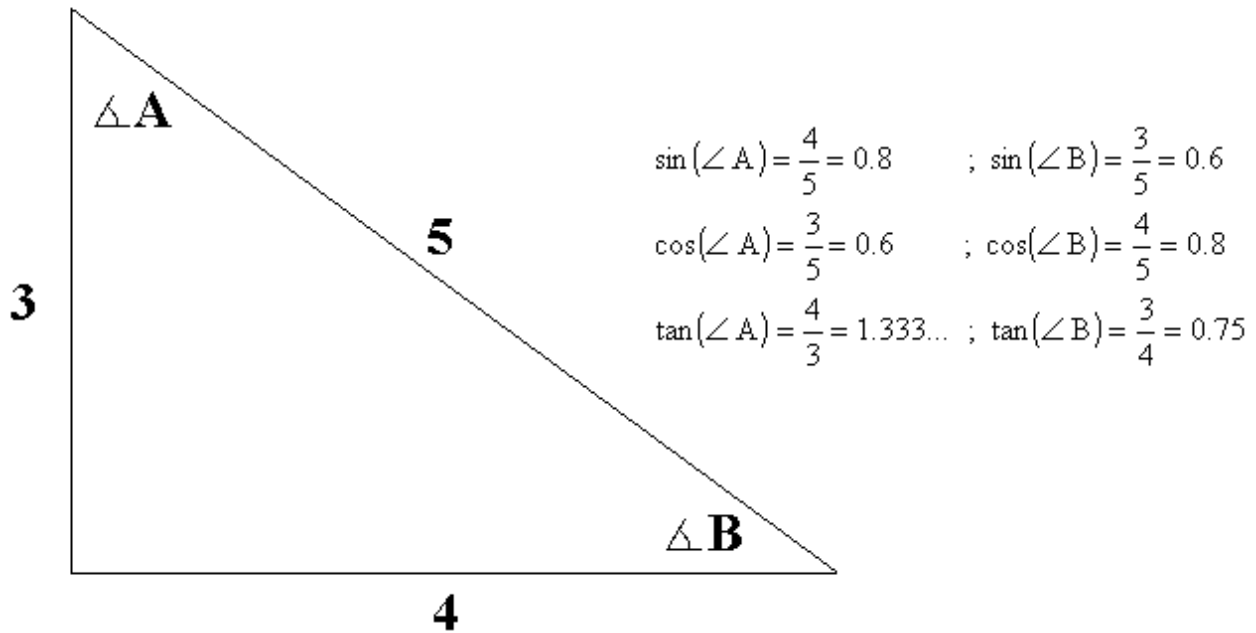
$$\sin(\text{acute } \angle) = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\cos(\text{acute } \angle) = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

$$\tan(\text{acute } \angle) = \frac{\text{opposite side}}{\text{adjacent side}}$$



Consider the “famous” 3, 4, 5 right triangle. The values of the three trig functions are computed below for both of the acute angles A and B.



To determine the values of these acute angles we need the “inverse” trig functions. A given acute angle uniquely determines the shape and thus the values of the ratios of the sides). So we can reverse this process and determine the acute angle from a given ratio of the sides. The angle is then a function of the ratio. These functions are most commonly designated as \sin^{-1} , \cos^{-1} , and \tan^{-1} . On most calculators they are accessed by using a **2nd**, **SHIFT**, or **INV** key in combination with the primary trig function key. Using the inverse trig functions, angle A in the 3, 4, 5 triangle can be calculated by the various methods illustrated below. The results are shown to the nearest ten thousandth decimal degree and to the nearest second.

$$A = \sin^{-1}(0.8) = 53.1301^\circ = 53^\circ 07' 48''$$

$$A = \cos^{-1}(0.6) = 53.1301^\circ = 53^\circ 07' 48''$$

$$A = \tan^{-1}\left(\frac{4}{3}\right) = 53.1301^\circ = 53^\circ 07' 48''$$

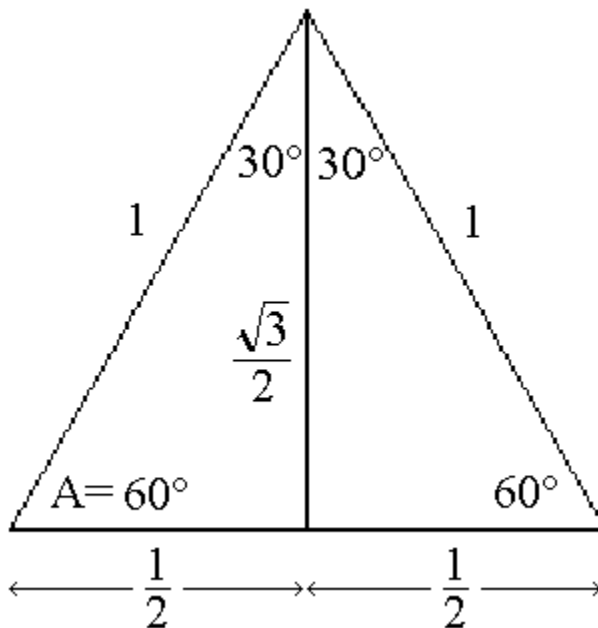
$$A = 90^\circ - B = 90^\circ - \sin^{-1}(0.6) = 90^\circ - 36.8699^\circ = 53.1301^\circ = 53^\circ 07' 48''$$

$$A = 90^\circ - B = 90^\circ - \cos^{-1}(0.8) = 90^\circ - 36.8699^\circ = 53.1301^\circ = 53^\circ 07' 48''$$

$$A = 90^\circ - B = 90^\circ - \tan^{-1}(0.75) = 90^\circ - 36.8699^\circ = 53.1301^\circ = 53^\circ 07' 48''$$

The following two examples using “special” triangles confirm that the inverse trig functions give the correct answer for the appropriate angle.

Consider an equilateral triangle of side 1. Dropping a perpendicular from the top vertex to the opposite side creates two right triangles with legs $\frac{1}{2}$ and $\sqrt{1 - \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$, and a hypotenuse of 1.



$$\sin(A) = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2} = 0.8660254$$

$$\cos(A) = \frac{\frac{1}{2}}{1} = \frac{1}{2} = 0.5$$

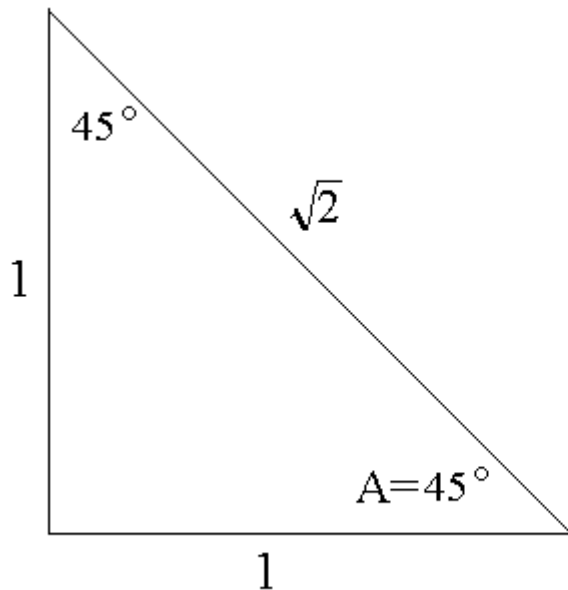
$$\tan(A) = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3} = 1.7320508$$

$$A = \sin^{-1}(0.8660254) = 60.0000^\circ$$

$$A = \cos^{-1}(0.5) = 60.0000^\circ$$

$$A = \tan^{-1}(1.7320508) = 60.0000^\circ$$

Now consider an isosceles right triangle with legs of length 1 and a hypotenuse of $\sqrt{1^2 + 1^2} = \sqrt{2}$.



$$\sin(A) = \frac{1}{\sqrt{2}} = 0.707106781187$$

$$\cos(A) = \frac{1}{\sqrt{2}} = 0.707106781187$$

$$\tan(A) = \frac{1}{1} = 1$$

$$A = \sin^{-1}(0.707106781187) = 45.0000^\circ$$

$$A = \cos^{-1}(0.707106781187) = 45.0000^\circ$$

$$A = \tan^{-1}(1) = 45.0000^\circ$$

Your Turn!!

1. Convert the following angles from decimal degrees to DMS notation.

DMS

$$36.35^\circ = \underline{\hspace{2cm}}$$

$$152.19^\circ = \underline{\hspace{2cm}}$$

$$262.25^\circ = \underline{\hspace{2cm}}$$

$$0.157^\circ = \underline{\hspace{2cm}}$$

2. Convert the following angles from DMS notation to decimal degrees with four decimal places.

Decimal Degree

$$15^\circ 42' 19'' = \underline{\hspace{2cm}}$$

$$121^\circ 37' 53'' = \underline{\hspace{2cm}}$$

$$1^\circ 16' 29'' = \underline{\hspace{2cm}}$$

$$3' 14'' = \underline{\hspace{2cm}}$$

For each angle give the values of the three trig functions to four decimal places.

$$\theta = 49.76^\circ$$

$$\theta = 16^\circ 11' 49''$$

3. $\cos \theta = \underline{\hspace{2cm}}$
 $\sin \theta = \underline{\hspace{2cm}}$
 $\tan \theta = \underline{\hspace{2cm}}$

4. $\cos \theta = \underline{\hspace{2cm}}$
 $\sin \theta = \underline{\hspace{2cm}}$
 $\tan \theta = \underline{\hspace{2cm}}$

5. Given that $\sin \theta = 0.55$ and $0^\circ < \theta < 90^\circ$, determine the following:

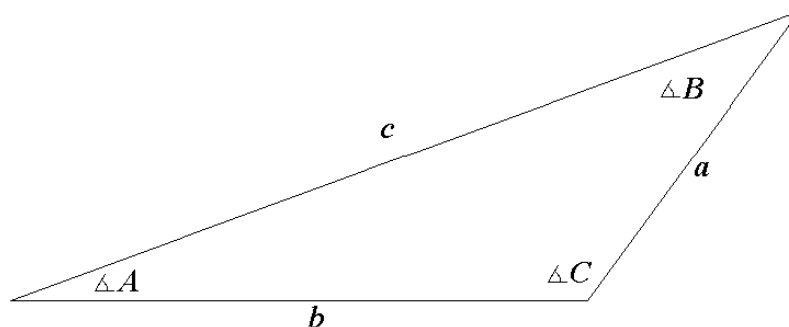
	DD (4 places)	DMS
$\theta =$	_____	_____
$\cos \theta =$	_____	
$\sin \theta =$	_____	
$\tan \theta =$	_____	

6. Given that $\tan \theta = 1.07$ and $0^\circ < \theta < 90^\circ$, determine the following:

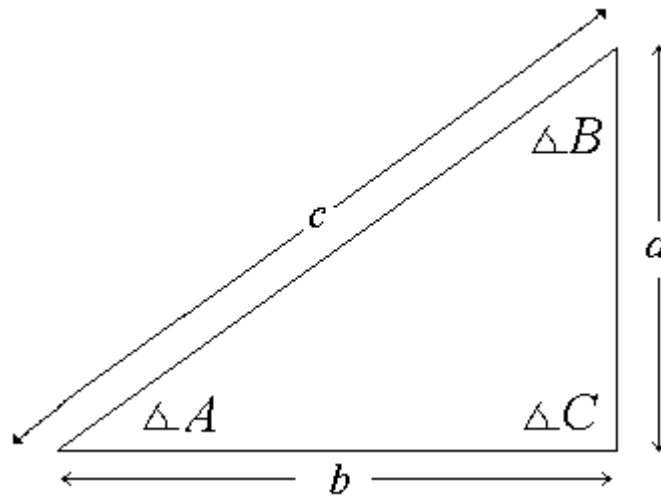
	DD (4 places)	DMS
$\theta =$	_____	_____
$\cos \theta =$	_____	
$\sin \theta =$	_____	
$\tan \theta =$	_____	

Section 7.2 Solving Right Triangles

With the three trig functions and their inverses we can solve for the missing information (lengths of sides and measure of angles) in any right triangle provided that we know either two sides or one side and one of the acute angles. To facilitate such problems the following notation will be used to describe any triangle. The **CAPITAL** letters A , B , and C will designate the three angles. The corresponding lower case letter will represent the side opposite to each angle. This is illustrated in the figure below.



For a right triangle $\angle C = 90^\circ$ and c is the hypotenuse.



In **any** triangle the largest side is opposite the largest angle and the smallest side is opposite the smallest angle.

The following three examples demonstrate the solution of right triangle problems.

Example 1

$$a = \underline{\hspace{2cm}} \qquad b = \underline{\hspace{2cm}} \qquad c = 7.56 \text{ m}$$

$$A = 36.5^\circ \qquad B = \underline{\hspace{2cm}} \qquad C = 90^\circ$$

Solution: Since angles A and B are complementary, $B = 90^\circ - 36.5^\circ = 53.5^\circ = 53^\circ 30'$.

From the definition of the sine,

$$\sin(36.5^\circ) = \frac{\text{opp}}{\text{hyp}} = \frac{a}{7.56 \text{ m}} \Rightarrow \text{solving for } a, \frac{a \cdot 7.56 \text{ m}}{7.56 \text{ m}} = 7.56 \text{ m} \cdot \sin(36.5^\circ) = 4.497 \text{ m}$$

so $a = 4.497 \text{ m}$.

The length of side b can now be determined using the cosine function.

$$\cos(36.5^\circ) = \frac{\text{adj}}{\text{hyp}} = \frac{b}{7.56 \text{ m}} \Rightarrow \text{solving for } b, \frac{b \cdot 7.56 \text{ m}}{7.56 \text{ m}} = 7.56 \text{ m} \cdot \cos(36.5^\circ) = 6.077 \text{ m}$$

so $b = 6.077 \text{ m}$.

There are two other methods of solution for b which use the calculated value of a and either the Pythagorean Theorem or the tangent function. However, any method which uses calculated values rather than the initial “given” values (the values of c , C , and A) risks unintended inaccuracy due to rounding errors. Since the solution shown above uses **only** the given information, it is the preferred method of solution.

Example 2

$$a = 12.0 \text{ ft} \qquad b = \underline{\hspace{2cm}} \qquad c = \underline{\hspace{2cm}}$$

$$A = \underline{\hspace{2cm}} \qquad B = 19^\circ 23' 39'' \qquad C = 90^\circ$$

Solution: Angle A is the complement of angle B , so

$$A = 90^\circ - B = 90^\circ - 19^\circ 23' 39'' = 89^\circ 59' 60'' - 19^\circ 23' 39'' = 70^\circ 36' 21''.$$

Next we use the tangent function to calculate the length of side b , then the cosine to calculate the hypotenuse, c .

$$19^\circ 23' 39'' = 19.39417^\circ$$

$$\tan(19.39417^\circ) = \frac{b}{12.0 \text{ ft}} \Rightarrow b = 12.0 \text{ ft} \cdot \tan(19.39417^\circ) = 4.224 \text{ ft}.$$

$$\cos(19.39417^\circ) = \frac{a}{c} = \frac{12.0 \text{ ft}}{c} \Rightarrow c \cdot \cos(19.39417^\circ) = \frac{12.0 \text{ ft} \cdot c}{c} \Rightarrow c = \frac{12.0 \text{ ft}}{\cos(19.39417^\circ)} = 12.722 \text{ ft}.$$

Again other solution methods can be used, but the approach given above solves for all unknowns solely in terms of the initial given information.

Example 3

$$a = 23.3 \text{ cm} \qquad b = \underline{\hspace{2cm}} \qquad c = 37.8 \text{ cm}$$

$$A = \underline{\hspace{2cm}} \qquad B = \underline{\hspace{2cm}} \qquad C = 90^\circ$$

Solution: First use the Pythagorean Theorem to calculate the length of side b , then use the inverse sine to calculate one of the missing acute angles.

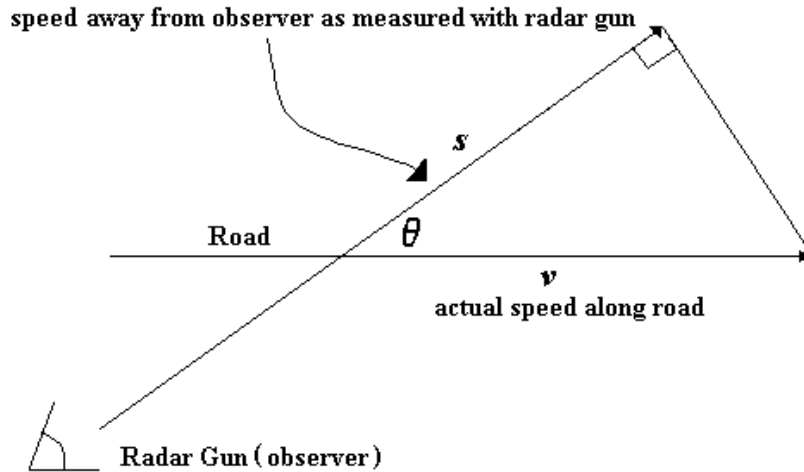
$$b = \sqrt{c^2 - a^2} = \sqrt{37.8^2 - 23.3^2} \text{ cm} = 29.7649 \text{ cm}.$$

$$\sin(A) = \frac{a}{c} = \frac{23.3}{37.8} = 0.61640 \Rightarrow A = \sin^{-1}(0.61640) = 38.0539^\circ = 38^\circ 03' 14''.$$

$$B = 90^\circ - A = 90^\circ - 38.0539^\circ = 51.9461^\circ = 51^\circ 56' 46''.$$

In the above examples **more** digits were reported in the computed answers than the precision of the given data warranted. This was done to illustrate the use of the calculator in solving these problems and to avoid round off discrepancies when comparing different approaches. In practice, the computed answers would be reported to a fewer number of digits.

In law enforcement the solution of right triangles is used to calculate the actual speed of a vehicle from the radar reading of its speed. This is often referred to as the “cosine angle error”. Consider the diagram shown below. The angle between the direction of the radar gun and the road along which the vehicle is moving is given as theta. The radar gun actually measures the “component” of the vehicle’s speed along its direction of sight. So unless theta is zero (i.e., unless the gun is aimed parallel to the road), the speed determined by radar is actually less than the actual speed of the vehicle.



Let s stand for the speed measured with the radar gun and let v be the actual speed of the vehicle along the road. Then from trigonometry we have the relations stated below.

$$s = v \cdot \cos(\theta) \quad \text{or} \quad v = \frac{s}{\cos(\theta)}$$

For example, if a the radar gun makes a 20 degree angle with the road and reads 85 mph, the actual speed of the car road is calculated as follows:

$$v = \frac{85 \text{ mph}}{\cos(20^\circ)} = 90.5 \text{ mph}$$

Similarly, if the car is travelling at 65 mph and the radar gun makes a 45 degree angle with the road, then the speed as read from the radar unit is given by

$$s = 65 \text{ mph} \cdot \cos(45^\circ) = 46.0 \text{ mph} .$$

Your Turn!!

Solve for the missing sides (3 digits) and angles (2 decimal places) in the following triangles. The notation is that the angle whose measure is specified by the capital letter is opposite the side whose length is specified by the lower case letter.

1. $a = 16.0 \text{ cm}$ $b = 12.0 \text{ cm}$ $c = \underline{\hspace{2cm}}$
 $A = \underline{\hspace{2cm}}$ $B = \underline{\hspace{2cm}}$ $C = 90.00^\circ$

2. $a = \underline{\hspace{2cm}}$ $b = \underline{\hspace{2cm}}$ $c = 5.00 \text{ cm}$
 $A = 40.00^\circ$ $B = \underline{\hspace{2cm}}$ $C = 90.00^\circ$

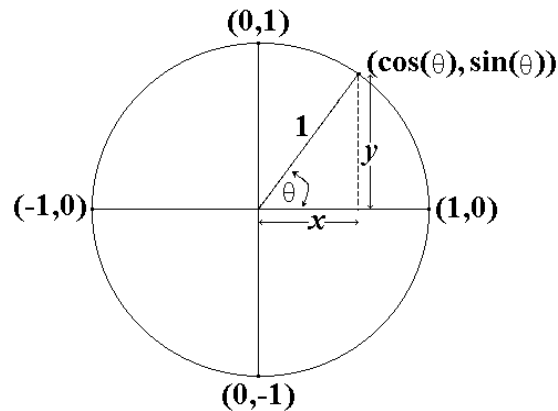
3. $a = 25.4 \text{ ft}$ $b = \underline{\hspace{2cm}}$ $c = \underline{\hspace{2cm}}$
 $A = 17.21^\circ$ $B = \underline{\hspace{2cm}}$ $C = 90.00^\circ$

4. A radar gun makes a 32 degree angle with a road and measures a speed of 59.4 mph. How fast is the car actually traveling?
5. If a radar gun is positioned at a 35 degree angle with respect to the road and the speed limit is 35 mph, what minimum radar reading means an actual speed more than 10 mph over the speed limit?
6. If a car's actual speed is 60 mph, fill in the following table. Give answers to the nearest tenth of a mile per hour.

Angle between direction of radar gun and road	Radar reading of car's speed
0°	
10°	
20°	
30°	
40°	
50°	
60°	
70°	
80°	
90°	

Section 7.3 The Law of Sines and the Law of Cosines

In order to apply the trigonometric functions to a wider variety of problems we need ways of solving oblique (i.e., non-right) triangles. The two tools that allow us to proceed are the Law of Sines and the Law of Cosines. However, we first need to extend the definition of the trig functions from acute angles to any angle. To do this we construct a circle of radius 1 centered at the origin of an $x - y$ set of coordinates. As shown below this circle crosses the axes at the points $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$.



From the origin construct a line segment that intersects the circle at a point with coordinates (x, y) . The angle θ is measured from the positive x axis to this segment. The trig functions are then defined as follows:

$$\cos(\theta) = \frac{x}{1} = x$$

$$\sin(\theta) = \frac{y}{1} = y$$

$$\tan(\theta) = \frac{y}{x} \text{ if } x \neq 0$$

If both x and y are positive θ is an acute angle and the definitions given above are just the results for a right triangle with adjacent leg x , opposite leg y , and a hypotenuse of 1. If either x or y is negative or zero, then θ is not an acute angle and we obtain an extension of our earlier definitions. For example, we now have the following results.

$$\cos(270^\circ) = 0$$

$$\cos(180^\circ) = -1$$

$$\cos(0^\circ) = 1$$

$$\cos(90^\circ) = 0$$

$$\sin(270^\circ) = -1$$

$$\sin(180^\circ) = 0$$

$$\sin(0^\circ) = 0$$

$$\sin(90^\circ) = 1$$

$$\tan(270^\circ) \text{ is undefined}$$

$$\tan(180^\circ) = 0$$

$$\tan(0^\circ) = 0$$

$$\tan(90^\circ) \text{ is undefined}$$

If an angle is negative, it is measured “down” from the x axis, i.e., the angle moves “clockwise” around the circle. For negative inputs, the inverse sine and inverse tangent functions will return an angle between negative ninety degrees and zero degrees, meaning that the angle is in that part of the coordinate system where x is positive and y is negative. On the other hand, for a negative input, the inverse cosine function will return an angle between positive ninety degrees and positive one hundred eighty degrees (that is, an obtuse angle). This means that such an angle is in that part of the coordinate system where x is negative and y is positive. **Note:** Even though the inverse cosine function can return an obtuse angle, the inverse sine **cannot** do so. Thus, we get the following results.

$$\cos^{-1}(-0.707106781) = 135^\circ$$

$$\sin^{-1}(-0.707106781) = -45^\circ$$

$$\tan^{-1}(-1) = -45^\circ$$

As the diagram below illustrates if

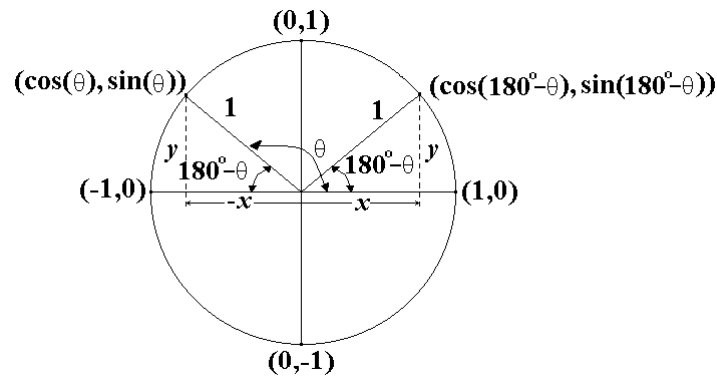
$$\cos(\theta) = -x \text{ and } \sin(\theta) = y, \text{ then}$$

$$\cos(180^\circ - \theta) = x \text{ and } \sin(180^\circ - \theta) = y$$

or

$$\cos(180^\circ - \theta) = -\cos(\theta)$$

$$\sin(180^\circ - \theta) = \sin(\theta).$$



Finally, by the Pythagorean Theorem, for any point with coordinates (x, y) on this circle, we have the equation.

$$x^2 + y^2 = 1$$

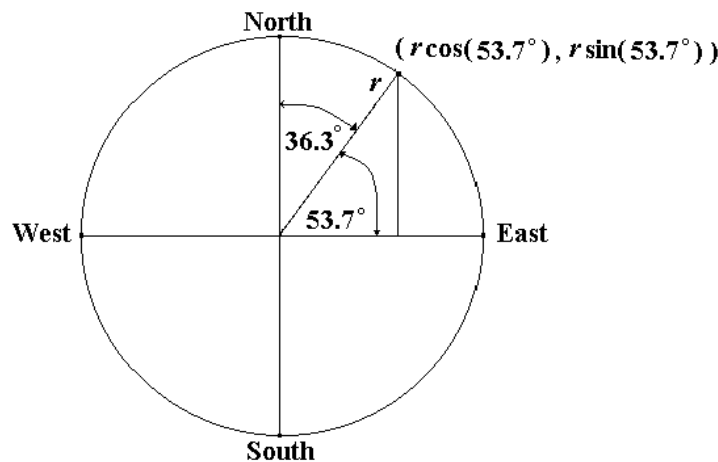
or for any angle θ

$$[\cos(\theta)]^2 + [\sin(\theta)]^2 = 1 .$$

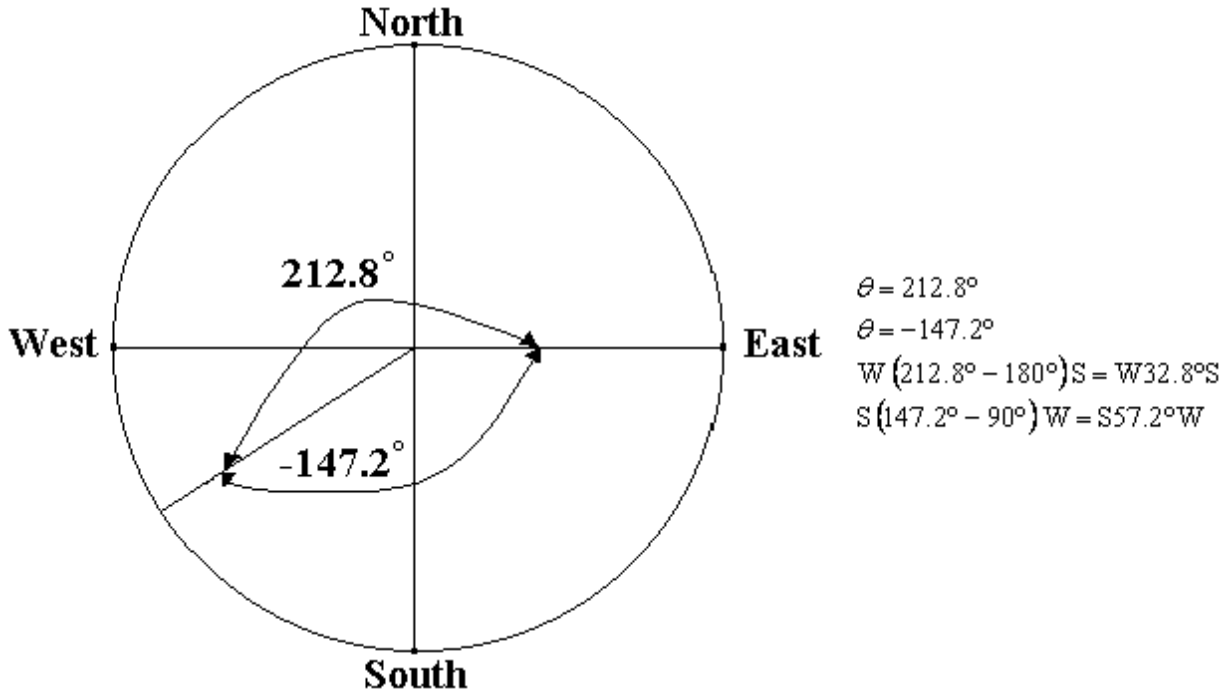
The use of trig functions defined for all angles is useful in navigation. The standard notation is to take North as the direction of the positive y axis and East as the direction of the positive x axis. If the angle theta is measured from East, then the positions east and north of the origin (the starting position) are given respectively by

$$x = r \cdot \cos(\theta) \quad \text{and} \quad y = r \cdot \sin(\theta)$$

Where r is the distance from the origin. If x is negative, the position is to the west and if y is negative the position is to the south. A special notation is also used to specify angles. For example, in the diagram below the orientation could be specified as a direction-angle-direction such as $E53.7^\circ N$ or $N36.3^\circ E$. This is also sometimes stated as 53.7 degrees north of east or 36.3 degrees east of north.



In the diagram below the orientation could be specified in any of the following equivalent ways.



For example, consider the following problem. If a plane flies $W18.6^\circ N$ at a speed of 347 mph for 33 minutes, how far west and north has the plane traveled?

Solution:

$$\theta = 180^\circ - 18.6^\circ = 161.4^\circ$$

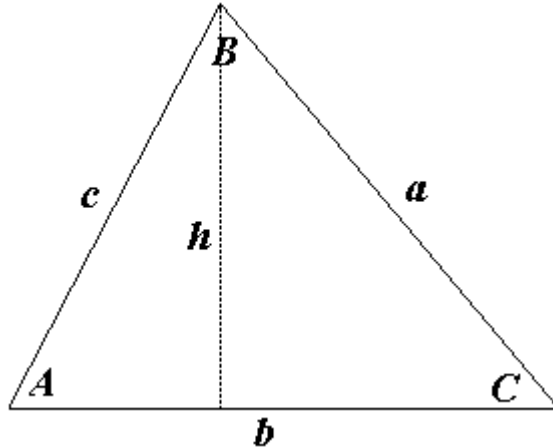
$$r = 347 \frac{\text{miles}}{\text{hr}} \cdot 33 \text{ min} \cdot \frac{1 \text{ hr}}{60 \text{ min}} = 191 \text{ miles}$$

$$x = r \cdot \cos(\theta) = -181 \text{ miles}$$

$$y = r \cdot \sin(\theta) = 60.9 \text{ miles}$$

So the plane has traveled a total distance of 191 miles. It is 181 miles west and 60.9 miles north of its original position.

The Law of Sines can be explained by the following argument. Consider an acute (i.e., the largest angle is less than a right angle) triangle. Drop a perpendicular of length h from the top vertex to the bottom base.



$$h = c \cdot \sin(A)$$

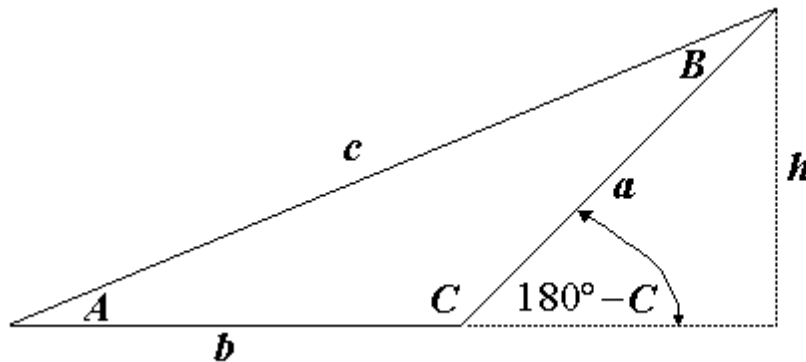
$$h = a \cdot \sin(C)$$

$$\text{So } c \cdot \sin(A) = a \cdot \sin(C)$$

$$\frac{c \cdot \sin(A)}{a \cdot c} = \frac{a \cdot \sin(C)}{a \cdot c}$$

$$\frac{\sin(A)}{a} = \frac{\sin(C)}{c}$$

Since the labeling of the sides and opposite angles is completely arbitrary this last ratio must also equal the ratio of the sine of angle B to the length of side b . If the triangle were obtuse (i.e., the largest angle is larger than a right angle) the same result is obtained as shown below.



$$h = c \cdot \sin(A)$$

$$h = a \cdot \sin(180^\circ - C)$$

$$\text{but, } \sin(180^\circ - C) = \sin(C)$$

$$\text{So } c \cdot \sin(A) = a \cdot \sin(C)$$

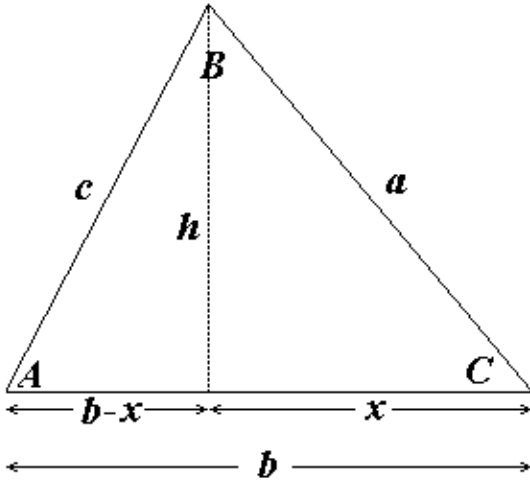
$$\frac{c \cdot \sin(A)}{a \cdot c} = \frac{a \cdot \sin(C)}{a \cdot c}$$

$$\frac{\sin(A)}{a} = \frac{\sin(C)}{c}$$

Thus, for **any** triangle we have that the ratio of the sine of any internal angle to the length of the opposite side is a constant for **that** specific triangle. The Law of Sines is summarized below.

$$\boxed{\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c} \quad \text{or} \quad \frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}}$$

To justify the Law of Cosines again consider the acute triangle shown. Drop a perpendicular of length h to the bottom base. Using the Pythagorean Theorem on both of the right triangles formed yields the following equations.



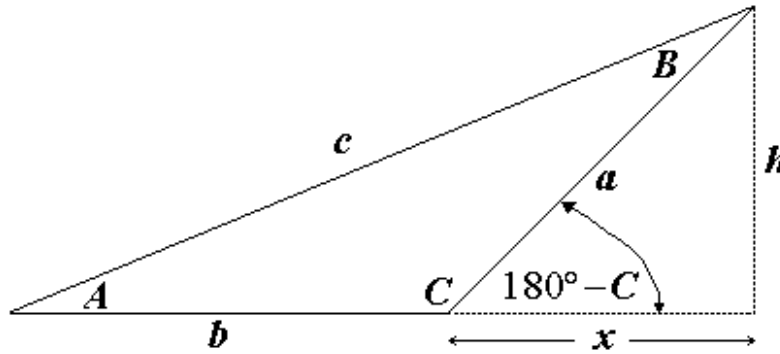
$$\begin{aligned}
 c^2 &= (b-x)^2 + h^2 \\
 (b-x)^2 &= (b-x) \cdot (b-x) \\
 &= (b-x) \cdot b + (b-x) \cdot (-x) \\
 &= b \cdot (b-x) + (-x) \cdot (b-x) \\
 &= b \cdot b + b \cdot (-x) + (-x) \cdot b + (-x) \cdot (-x) \\
 &= b^2 - b \cdot x - x \cdot b + x^2 = b^2 - 2b \cdot x + x^2
 \end{aligned}$$

But, $x = a \cdot \cos(C)$ and $x^2 + h^2 = a^2$

So $c^2 = b^2 - 2b \cdot x + x^2 + h^2 = b^2 - 2b \cdot a \cdot \cos(C) + a^2$

$$c^2 = a^2 + b^2 - 2a \cdot b \cdot \cos(C)$$

Again since the labeling of sides and angles was arbitrary, similar results for the square of sides a and b must also be true. If the triangle is obtuse, we still obtain the same results as shown below.



$$\begin{aligned}
 c^2 &= (b+x)^2 + h^2 \\
 (b+x)^2 &= (b+x)(b+x) = b(b+x) + x(b+x) \\
 &= b \cdot b + b \cdot x + b \cdot x + x \cdot x = b^2 + 2b \cdot x + x^2
 \end{aligned}$$

but, $x = a \cdot \cos(180^\circ - C) = -a \cos(C)$ and $x^2 + h^2 = a^2$.

So, $c^2 = b^2 + 2bx + x^2 + h^2 = b^2 + 2b \cdot (-a \cos(C)) + a^2$

$$= a^2 + b^2 - 2a \cdot b \cos(C)$$

In summary, the Law of Cosines states that for **any** triangle the length of any side squared equals the sum of the squares of the other two sides minus twice the product of the other two sides with the cosine of the opposite angle. The Law of Cosines generalizes the Pythagorean Theorem to any kind of triangle, not just those with a right angle.

$$\begin{aligned}
 c^2 &= a^2 + b^2 - 2a \cdot b \cdot \cos(C) \Rightarrow c = \sqrt{a^2 + b^2 - 2a \cdot b \cdot \cos(C)} \\
 b^2 &= a^2 + c^2 - 2a \cdot c \cdot \cos(B) \Rightarrow b = \sqrt{a^2 + c^2 - 2a \cdot c \cdot \cos(B)} \\
 a^2 &= b^2 + c^2 - 2b \cdot c \cdot \cos(A) \Rightarrow a = \sqrt{b^2 + c^2 - 2b \cdot c \cdot \cos(A)}
 \end{aligned}$$

Rearranging these formulas to solve for the relevant angles gives the following results.

$$c^2 = a^2 + b^2 - 2a \cdot b \cdot \cos(C) \Rightarrow c^2 - a^2 - b^2 = -2a \cdot b \cdot \cos(C)$$

$$\frac{c^2 - a^2 - b^2}{-2a \cdot b} = \frac{-2a \cdot b \cdot \cos(C)}{-2a \cdot b}$$

$$\cos(C) = -\frac{c^2 - a^2 - b^2}{2a \cdot b} = \frac{(-1) \cdot (c^2 - a^2 - b^2)}{2a \cdot b} = \frac{a^2 + b^2 - c^2}{2a \cdot b}$$

$\cos(C) = \frac{a^2 + b^2 - c^2}{2a \cdot b} \Rightarrow C = \cos^{-1}\left(\frac{(a^2 + b^2 - c^2)}{(2a \cdot b)}\right)$
$\cos(B) = \frac{a^2 + c^2 - b^2}{2a \cdot c} \Rightarrow B = \cos^{-1}\left(\frac{(a^2 + c^2 - b^2)}{(2a \cdot c)}\right)$
$\cos(A) = \frac{b^2 + c^2 - a^2}{2b \cdot c} \Rightarrow A = \cos^{-1}\left(\frac{(b^2 + c^2 - a^2)}{(2b \cdot c)}\right)$

The unneeded parentheses were added to the formulas for the sides a , b , c and the angles A , B and C to emphasize that both the square root symbol and the fraction bar act as implied grouping symbols. Failure to provide these parentheses on a scientific calculator will result in a wrong answer.

Since the inverse cosine function returns angles in the range 0° to 180° , the Law of Cosines can be used to find **any** angle of a triangle when the lengths of all three sides are known. In contrast, since the inverse sine function can't return angles between 90° to 180° , the Law of Sines will be unable to directly compute the measure of an obtuse angle in a triangle.

Your Turn!!

A plane travels W35°N (i.e., 35 degrees north of west) for 15 minutes at 350 mph.

1. In these 15 minutes how far did the plane travel?
2. In these 15 minutes how far west did the plane travel?
3. In these 15 minutes how far north did the plane travel?

4. Given that $\sin \theta = 0.35$ and $90^\circ < \theta < 180^\circ$, determine the following:

DD (4 places)	DMS
$\theta =$ _____	_____
$\cos \theta =$ _____	
$\sin \theta =$ _____	
$\tan \theta =$ _____	

5. Given that $\cos \theta = -0.55$ and $90^\circ < \theta < 180^\circ$, determine the following:

	DD (4 places)	DMS
$\theta =$	_____	_____
$\cos \theta =$	_____	
$\sin \theta =$	_____	
$\tan \theta =$	_____	

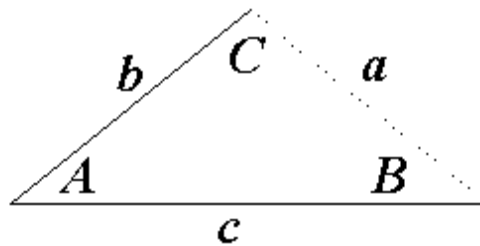
6. Given that $\tan \theta = 1.25$ and $180^\circ < \theta < 270^\circ$, determine the following:

	DD (4 places)	DMS
$\theta =$	_____	_____
$\cos \theta =$	_____	
$\sin \theta =$	_____	
$\tan \theta =$	_____	

Section 7.4 Solving Oblique Triangles

With the sine and cosine functions and their inverses we can solve for the three missing items of information (lengths of sides or measure of angles) in **any** triangle provided that we have the three relevant inputs. The three relevant inputs are based on the congruency criteria discussed in Chapter 6. Specifically, these are SAS, ASA, and SSS. The procedure for solving a triangle in each case is presented below.

1. SAS



Given two sides b and c and the angle A between these sides:

A. Use the Law of Cosines to calculate the length of the side opposite the given angle.

$$a^2 = b^2 + c^2 - 2b \cdot c \cdot \cos(A) \Rightarrow a = \sqrt{b^2 + c^2 - 2b \cdot c \cdot \cos(A)}$$

B. Use the Law of Sines to calculate the measure of the angle opposite to the smaller of sides a and b . This is done to avoid trying to compute an obtuse angle using the inverse sine function. Then calculate the third missing angle by subtracting the sum of the two known angles from 180° .

$$\text{If } b > c \quad \frac{\sin(C)}{c} = \frac{\sin(A)}{a} \Rightarrow \sin(C) = \frac{c \cdot \sin(A)}{a} \Rightarrow C = \sin^{-1}\left(\frac{c \cdot \sin(A)}{a}\right)$$

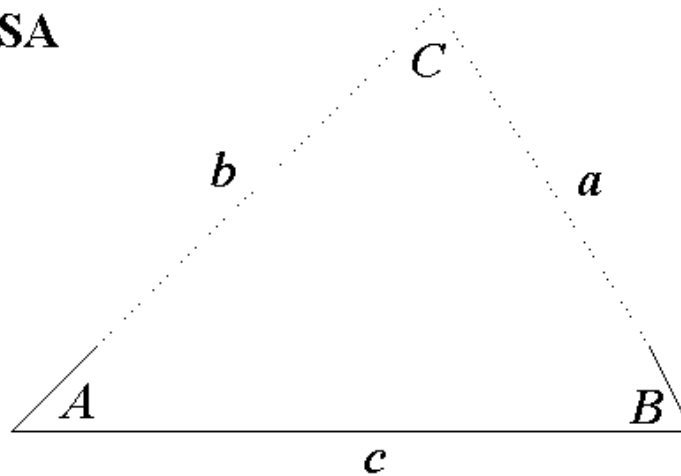
$$B = 180^\circ - A - C$$

$$\text{If } c > b \quad \frac{\sin(B)}{b} = \frac{\sin(A)}{a} \Rightarrow \sin(B) = \frac{b \cdot \sin(A)}{a} \Rightarrow B = \sin^{-1}\left(\frac{b \cdot \sin(A)}{a}\right)$$

$$C = 180^\circ - A - B$$

Note: In the formulas above we used the version of the Law of Sines with the sines of the unknown angles in the numerator. This is not at all necessary, but it does make the algebra of solving for the unknown angles a little easier to follow.

2. ASA



Given two angles A and B and c the side between them:

- A. Calculate the missing angle by subtracting the sum of the two given angles from 180 degrees.
 B. Use the Law of Sines to calculate the two missing sides.

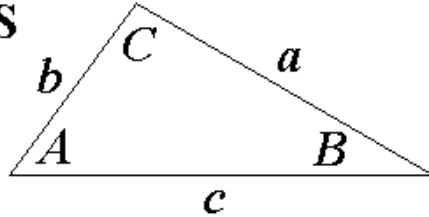
$$C = 180^\circ - A - B$$

$$\frac{a}{\sin(A)} = \frac{c}{\sin(C)} \Rightarrow a = \frac{c \cdot \sin(A)}{\sin(C)}$$

$$\frac{b}{\sin(B)} = \frac{c}{\sin(C)} \Rightarrow b = \frac{c \cdot \sin(B)}{\sin(C)}$$

Note: In the formulas above we used the version of the Law of Sines with the unknown sides in the numerator.

3. SSS



Given the three sides, a, b, c , determine the three angles A, B, C :

- A. Use the Law of Cosines to calculate the measure of the angle opposite to the longest side. This is the only angle in the triangle that could be obtuse. Hence this angle can always be calculated using the inverse cosine function, but may not be directly computable using the inverse sine function.

$$C = \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2a \cdot b}\right)$$

- B. Use the Law of Sines to calculate the measure of one of the remaining two unknown angles. Then calculate the third missing angle by subtracting the sum of the two known angles from 180 degrees.

$$\frac{\sin(A)}{a} = \frac{\sin(C)}{c} \Rightarrow \sin(A) = \frac{a \cdot \sin(C)}{c} \Rightarrow A = \sin^{-1}\left(\frac{a \cdot \sin(C)}{c}\right)$$

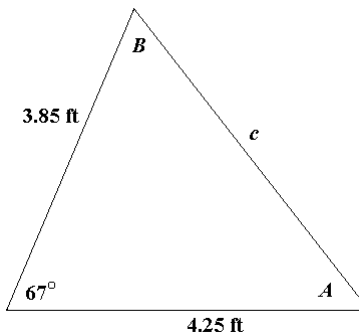
$$B = 180^\circ - A - C$$

Example 1

Given a triangle with the following information, solve for the missing side and angles.

$$\begin{array}{lll} a & = & 3.85 \text{ ft} \\ b & = & 4.25 \text{ ft} \\ c & = & \underline{\hspace{2cm}} \\ A & = & \underline{\hspace{2cm}} \\ B & = & \underline{\hspace{2cm}} \\ C & = & 67^\circ \end{array}$$

Solution: This is a SAS problem. First we calculate the missing side c , then the angles A and B .



$$c = \sqrt{a^2 + b^2 - 2a \cdot b \cdot \cos(C)} = \sqrt{3.85^2 + 4.25^2 - 2 \times 3.85 \times 4.25 \times \cos(67^\circ)} \text{ ft} = \sqrt{20.098} \text{ ft}$$

$$c = 4.483 \text{ ft}$$

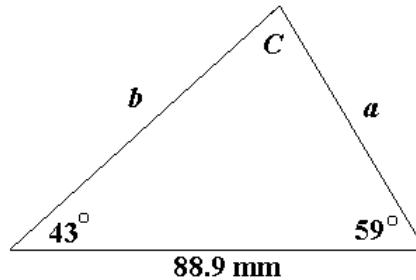
$$\frac{\sin(A)}{3.85 \text{ ft}} = \frac{\sin(67^\circ)}{4.483 \text{ ft}} \Rightarrow \sin(A) = \frac{3.85 \text{ ft} \times \sin(67^\circ)}{4.483 \text{ ft}} = 0.7905 \Rightarrow A = \sin^{-1}(0.7905) = 52.23^\circ$$

$$B = 180^\circ - A - C = 180^\circ - 52.23^\circ - 67^\circ = 60.77^\circ$$

Example 2

Given a triangle with the following information, solve for the missing sides and angle.

$$\begin{array}{lll} a & = & \underline{\hspace{2cm}} \\ b & = & \underline{\hspace{2cm}} \\ c & = & 88.9 \text{ mm} \\ A & = & 43^\circ \\ B & = & 59^\circ \\ C & = & \underline{\hspace{2cm}} \end{array}$$



Solution: This is an ASA problem. First we calculate the missing angle C , then the sides a and b .
 $C = 180^\circ - 43^\circ - 59^\circ = 78^\circ$

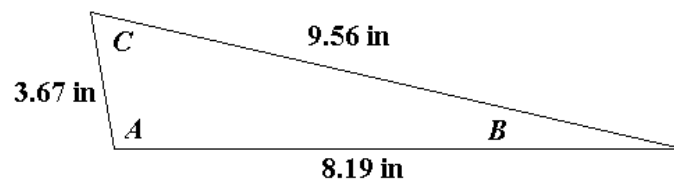
$$\frac{a}{\sin(A)} = \frac{c}{\sin(C)} \Rightarrow a = \frac{c \sin(A)}{\sin(C)} = \frac{88.9 \text{ mm} \times \sin(43^\circ)}{\sin(78^\circ)} = 61.98 \text{ mm}.$$

$$\frac{b}{\sin(B)} = \frac{c}{\sin(C)} \Rightarrow b = \frac{c \sin(B)}{\sin(C)} = \frac{88.9 \text{ mm} \times \sin(59^\circ)}{\sin(78^\circ)} = 77.90 \text{ mm}$$

Example 3

Given a triangle with the following information, solve for the three missing angles.

$$\begin{array}{lll} a & = & 9.56 \text{ in} \\ b & = & 3.67 \text{ in} \\ c & = & 8.19 \text{ in} \\ A & = & \underline{\hspace{2cm}} \\ B & = & \underline{\hspace{2cm}} \\ C & = & \underline{\hspace{2cm}} \end{array}$$



Solution: This is an SSS problem. First we calculate the missing angle A (the largest angle), then the angles C and B .

$$A = \cos^{-1}\left(\frac{(b^2 + c^2 - a^2)}{(2 \cdot b \cdot c)}\right) = \cos^{-1}\left(\frac{(3.67^2 + 8.19^2 - 9.56^2)}{(2 \times 3.67 \times 8.19)}\right) = \cos^{-1}(-0.18046) = 100.40^\circ$$

$$\frac{\sin(C)}{c} = \frac{\sin(A)}{a} \Rightarrow \sin(C) = \frac{c \sin(A)}{a} = \frac{8.19 \text{ in} \times \sin(100.40^\circ)}{9.56 \text{ in}} = 0.84263$$

$$C = \sin^{-1}(0.84263) = 57.42^\circ$$

$$B = 180^\circ - A - C = 180^\circ - 100.40^\circ - 57.42^\circ = 22.18^\circ$$

Note: If first we had calculated either angle B or C from the Law of Cosines and then had attempted to calculate angle A using the Law of Sines, we would have instead obtained A 's supplement. The reason for this is again the fact that for positive input, the inverse sine function always returns an angle between zero and ninety degrees. This illustrates the need to set up the calculation in such a way so that the Law of Sines is not used to calculate the angle opposite to the longest side. Such an **incorrect** solution is shown below.

$$B = \cos^{-1}\left(\frac{(a^2 + c^2 - b^2)}{(2 \cdot a \cdot c)}\right) = \cos^{-1}\left(\frac{(9.56^2 + 8.19^2 - 3.67^2)}{(2 \times 9.56 \times 8.19)}\right) = \cos^{-1}(0.92597) = 22.18^\circ$$

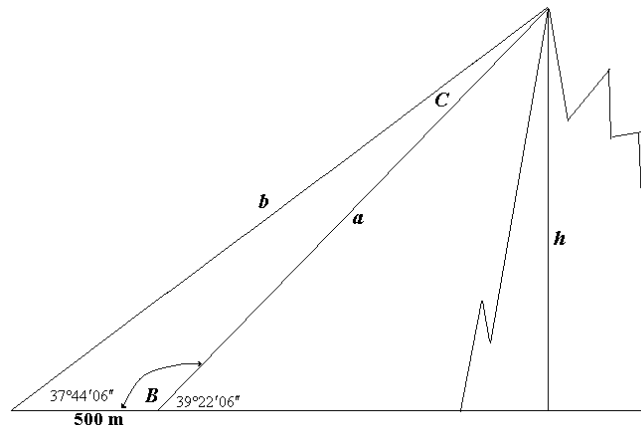
$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} \Rightarrow \sin(A) = \frac{a \sin(B)}{b} = \frac{9.56 \text{ in} \times \sin(22.18^\circ)}{3.67 \text{ in}} = 0.98358$$

$$\text{So, "A"} = \sin^{-1}(0.98358) = 79.60^\circ = 180^\circ - 100.40^\circ$$

Using this flawed procedure we obtained the supplement of the correct answer. Since $\sin(79.60^\circ) = \sin(100.40^\circ) = 0.98358$, the inverse sine function gave the “wrong” answer.

Example 4

An explorer is taking measurements on a distant mountain peak. She initially measures an angle of elevation of $37^\circ 44' 06''$. She then advances 500 meters closer and measures a new angle of elevation of $39^\circ 22' 06''$. How high is the mountain peak above the horizontal of the explorer?



Solution: Angles of elevation are measured from the horizontal “up” to a distant object. In a similar way angles of depression are measured from the horizontal “down” to a distant object. To determine the height h , we need to determine either one of the lengths a or b and use right triangle trigonometry. The determination of a or b means solving an ASA triangle. First we calculate the angle C , then use the Law of Sines to calculate either a or b .

$$A = 37^{\circ}44'06'' = 37.735^{\circ}$$

$$B = 180^{\circ} - 39^{\circ}22'06'' = 140^{\circ}37'54'' = 140.63167^{\circ}$$

$$C = 180^{\circ} - (A + B) = 180^{\circ} - 178^{\circ}22'' = 1^{\circ}38'' = 1.6333^{\circ}$$

$$\frac{a}{\sin(A)} = \frac{c}{\sin(C)} \Rightarrow a = \frac{c \sin(A)}{\sin(C)} = \frac{500\text{m} \times \sin(37.735^{\circ})}{\sin(1.6333^{\circ})} = 10,736. \text{m}$$

$$\text{So, } h = a \cdot \sin(39.3683^{\circ}) = 10,736\text{m} \times \sin(39.3683^{\circ}) = 6810. \text{m}$$

or

$$\frac{b}{\sin(B)} = \frac{c}{\sin(C)} \Rightarrow b = \frac{c \sin(B)}{\sin(C)} = \frac{500\text{m} \times \sin(140.6317^{\circ})}{\sin(1.6333^{\circ})} = 11,127. \text{m}$$

$$\text{So, } h = b \cdot \sin(37.735^{\circ}) = 11,127\text{m} \times \sin(37.735^{\circ}) = 6810. \text{m}$$

Your Turn!!

Solve for the missing sides (3 digits) and angles (2 decimal places) in the following triangles. The notation is that the angle whose measure is specified by the capital letter is opposite the side whose length is specified by the lower case letter.

1. $a = 17.2 \text{ cm}$ $b = 21.3 \text{ cm}$ $c = \underline{\hspace{2cm}}$
 $A = \underline{\hspace{2cm}}$ $B = \underline{\hspace{2cm}}$ $C = 45.00^{\circ}$

2. $a = 11.1 \text{ in}$ $b = 21.5 \text{ in}$ $c = 16.2 \text{ in}$
 $A = \underline{\hspace{2cm}}$ $B = \underline{\hspace{2cm}}$ $C = \underline{\hspace{2cm}}$

3. $a = \underline{\hspace{2cm}}$ $b = \underline{\hspace{2cm}}$ $c = 14.3 \text{ ft}$
 $A = 50.00^{\circ}$ $B = 30.00^{\circ}$ $C = \underline{\hspace{2cm}}$

4. A geologist sights a distant hilltop with an angle of elevation of 23 degrees 37 minutes. She then advances 1000 m closer to the mountain and measures a new angle of elevation of 29 degrees 10 minutes. How high is the hilltop above the level of the geologist?

5. Two airplanes leave the same airport at the same time. The planes are travelling 400 mph and 360 mph respectively. After one and one half-hours the planes are 200 miles apart. To two decimal places what is the angle between the airplanes' courses of flight?

Chapter 7 Practice Exam

1. Convert the following angles from decimal degrees to DMS notation and from DMS to decimal degrees to the nearest ten-thousandth.

Decimal Degree	DMS
$11^{\circ}30'$ = _____	41.18° = _____
$83^{\circ}19'10''$ = _____	129.25° = _____
$50'50''$ = _____	41.33° = _____
$42^{\circ}15'30''$ = _____	0.022° = _____

2. For each angle give the values of the three trig functions to four decimal places.

$\theta = 52^{\circ}16'49''$	$\theta = 20.25^{\circ}$
$\cos \theta$ = _____	$\cos \theta$ = _____
$\sin \theta$ = _____	$\sin \theta$ = _____
$\tan \theta$ = _____	$\tan \theta$ = _____

3. Given that $\sin \theta = 0.25$ and $0^{\circ} < \theta < 90^{\circ}$, determine the following:

DD (4 places)	DMS
θ = _____	_____
$\cos \theta$ = _____	
$\sin \theta$ = _____	
$\tan \theta$ = _____	

4. Given that $\tan \theta = 0.85$ and $0^{\circ} < \theta < 90^{\circ}$, determine the following:

DD (4 places)	DMS
θ = _____	_____
$\cos \theta$ = _____	
$\sin \theta$ = _____	
$\tan \theta$ = _____	

5. Given that $\cos \theta = -0.85$ and $90^{\circ} < \theta < 180^{\circ}$, determine the following:

DD (4 places)	DMS
θ = _____	_____
$\cos \theta$ = _____	
$\sin \theta$ = _____	
$\tan \theta$ = _____	

6. Given that $\tan \theta = 0.75$ and $180^\circ < \theta < 270^\circ$, determine the following:

	DD (4 places)	DMS
$\theta =$	_____	_____
$\cos \theta =$	_____	
$\sin \theta =$	_____	
$\tan \theta =$	_____	

Solve for the missing sides (3 digits) and angles (2 decimal places) in the following triangles. The notation is that the angle whose measure is specified by the capital letter is opposite the side whose length is specified by the lower case letter.

7. $a = 23.0$ cm $b = 18.5$ cm $c =$ _____
 $A =$ _____ $B =$ _____ $C = 90.00^\circ$

8. $a =$ _____ $b =$ _____ $c = 86.7$ ft
 $A = 23.75^\circ$ $B =$ _____ $C = 90.00^\circ$

9. $a = 3.68$ m $b =$ _____ $c =$ _____
 $A = 55.86^\circ$ $B =$ _____ $C = 90.00^\circ$

10. $a = 25.0$ cm $b = 33.6$ cm $c =$ _____
 $A =$ _____ $B =$ _____ $C = 120.00^\circ$

11. $a = 42.0$ in $b =$ _____ $c =$ _____
 $A = 38.41^\circ$ $B =$ _____ $C = 91.50^\circ$

12. $a = 19.6$ cm $b = 28.6$ cm $c = 43.2$ cm
 $A =$ _____ $B =$ _____ $C =$ _____

13. A radar gun makes a 45 degree angle with a road and measures a speed of 64.8 mph. How fast is the car actually traveling?

14. Two airplanes leave the same airport at the same time. The planes are traveling 500 mph and 480 mph respectively. After one half-hour the planes are 160 miles apart. To two decimal places what is the angle between the planes' lines of flight.

Chapter 7 Answer Section

Section 7.1 Solution Set:

1.	Decimal Degree	2.	DMS	3.	
	$15^{\circ}42'19'' = 15.7053^{\circ}$		$36.35^{\circ} = 36^{\circ}21'$		$\theta = 49.76^{\circ}$
	$121^{\circ}37'53'' = 121.6314^{\circ}$		$152.19^{\circ} = 152^{\circ}11'24''$		$\cos\theta = 0.6460$
	$1^{\circ}16'29'' = 1.2747^{\circ}$		$262.25^{\circ} = 262^{\circ}15'$		$\sin\theta = 0.7633$
	$3'14'' = 0.0539^{\circ}$		$0.157^{\circ} = 0^{\circ}9'25''$		$\tan\theta = 1.1817$

4.		5.	DD (4 places)	DMS
	$\theta = 16^{\circ}11'49''$		$\theta = 33.3670^{\circ} =$	$33^{\circ}22'01''$
	$\cos\theta = 0.9603$		$\cos\theta = 0.8352$	
	$\sin\theta = 0.2789$		$\sin\theta = 0.5500$	
	$\tan\theta = 0.2905$		$\tan\theta = 0.6586$	

6.	DD (4 places)	DMS
	$\theta = 46.9368^{\circ} =$	$46^{\circ}56'12''$
	$\cos\theta = 0.6828$	
	$\sin\theta = 0.7306$	
	$\tan\theta = 1.0700$	

Section 7.2 Solution Set:

Answers:

1.	$a = 16.0$ cm	$b = 12.0$ cm	$c = 20.0$ cm
	$A = 53.13^{\circ}$	$B = 36.87^{\circ}$	$C = 90.00^{\circ}$
2.	$a = 3.21$ cm	$b = 3.83$ cm	$c = 5.00$ cm
	$A = 40.00^{\circ}$	$B = 50.00^{\circ}$	$C = 90.00^{\circ}$
3.	$a = 25.4$ ft	$b = 82.0$ ft	$c = 85.8$ ft
	$A = 17.21^{\circ}$	$B = 72.79^{\circ}$	$C = 90.00^{\circ}$

4. 70.0 mph ; 5. 36.9 mph .

6.

Angle between direction of radar gun and road	Radar reading of car's speed
0°	60.0 mph
10°	59.1 mph
20°	56.4 mph
30°	52.0 mph
40°	46.0 mph
50°	38.6 mph
60°	30.0 mph
70°	20.5 mph
80°	10.4 mph
90°	0.0 mph

Section 7.3 Solution Set:

1. 87.5 miles ; 2. 71.7 miles ; 3. 50.2 miles

4.

DD (4 places)	DMS
$\theta = 159.5127^\circ$	$159^\circ 30' 46''$
$\cos \theta = -0.9367$	
$\sin \theta = 0.35$	
$\tan \theta = -0.3736$	

5.

DD (4 places)	DMS
$\theta = 123.3670^\circ$	$123^\circ 22' 1''$
$\cos \theta = -0.55$	
$\sin \theta = 0.8352$	
$\tan \theta = -1.5185$	

6.

DD (4 places)	DMS
$\theta = 231.3402^\circ$	$231^\circ 20' 25''$
$\cos \theta = -0.6247$	
$\sin \theta = -0.7809$	
$\tan \theta = 1.2500$	

Section 7.4 Solution Set:

1.	$a = 17.2 \text{ cm}$	$b = 21.3 \text{ cm}$	$c = 15.2 \text{ cm}$
	$A = 53.08^\circ$	$B = 81.92^\circ$	$C = 45.00^\circ$
2.	$a = 11.1 \text{ in}$	$b = 21.5 \text{ in}$	$c = 16.2 \text{ in}$
	$A = 30.29^\circ$	$B = 102.30^\circ$	$C = 47.41^\circ$

$$3. \quad \begin{array}{lll} a = 11.1 \text{ ft} & b = 7.26 \text{ ft} & c = 14.3 \text{ ft} \\ A = 50.00^\circ & B = 30.00^\circ & C = 100^\circ \end{array}$$

$$4. \quad 2019 \text{ m} ; 5. \quad 19.30^\circ = 19^\circ 17'$$

Chapter 7 Practice Exam Solution Set:

1.

Decimal Degree	DMS
$11^\circ 30'$ = 11.5000°	41.18° = 41°10'48"
$83^\circ 19' 10''$ = 83.3194°	129.25° = 129°15'00"
$50' 50''$ = 0.8472°	41.33° = 41°19'48"
$42^\circ 15' 30''$ = 42.0108°	0.022° = 0°01'19"

2.

$\theta = 52^\circ 16' 49''$	$\theta = 20.25^\circ$
$\cos \theta = 0.6118$	$\cos \theta = 0.9382$
$\sin \theta = 0.7910$	$\sin \theta = 0.3462$
$\tan \theta = 1.2929$	$\tan \theta = 0.3689$

3.

DD (4 places)	DMS
$\theta = 14.4775^\circ$	$14^\circ 28' 39''$
$\cos \theta = 0.9682$	
$\sin \theta = 0.2500$	
$\tan \theta = 0.2582$	

4.

DD (4 places)	DMS
$\theta = 40.3645^\circ$	$40^\circ 21' 52''$
$\cos \theta = 0.7619$	
$\sin \theta = 0.6476$	
$\tan \theta = 0.8500$	

5.

DD (4 places)	DMS
$\theta = 148.2117^\circ$	$148^\circ 12' 42''$
$\cos \theta = -0.8500$	
$\sin \theta = 0.5268$	
$\tan \theta = -0.6197$	

6.

DD (4 places)	DMS
$\theta = 216.8699^\circ$	$216^\circ 52' 12''$
$\cos \theta = -0.8000$	
$\sin \theta = -0.6000$	
$\tan \theta = 0.7500$	

$$7. \quad \begin{array}{lll} a = 23.0 \text{ cm} & b = 18.5 \text{ cm} & c = 29.5 \text{ cm} \\ A = 51.19^\circ & B = 38.81^\circ & C = 90.00^\circ \end{array}$$

- | | | | |
|-----|------------------------------------|-------------------------------------|-------------------------------------|
| 8. | $a = 34.9$ ft
$A = 23.75^\circ$ | $b = 79.4$ ft
$B = 66.25^\circ$ | $c = 86.7$ ft
$C = 90.00^\circ$ |
| 9. | $a = 3.68$ m
$A = 55.86^\circ$ | $b = 2.50$ m
$B = 34.14^\circ$ | $c = 4.45$ m
$C = 90.00^\circ$ |
| 10. | $a = 25.0$ cm
$A = 25.16^\circ$ | $b = 33.6$ cm
$B = 34.84^\circ$ | $c = 50.9$ cm
$C = 120.00^\circ$ |
| 11. | $a = 42.0$ in
$A = 38.41^\circ$ | $b = 43.53$ in
$B = 40.09^\circ$ | $c = 66.2$ in
$C = 101.50^\circ$ |
| 12. | $a = 19.6$ cm
$A = 21.44^\circ$ | $b = 28.6$ cm
$B = 32.23^\circ$ | $c = 43.2$ cm
$C = 126.33^\circ$ |
| 13. | 91.6 mph. | 14. 38.05° | |