The Triple Vector Cross Product Or the "Bac Cab" Theorem

Consider

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_x & a_y & a_z\\ \left(\vec{B}\times\vec{C}\right)_x & \left(\vec{B}\times\vec{C}\right)_y & \left(\vec{B}\times\vec{C}\right)_z\end{vmatrix}.$$

$$=\hat{i}\left[a_y\left(\vec{B}\times\vec{C}\right)_z-\left(\vec{B}\times\vec{C}\right)_y a_z\right]+\hat{j}\left[a_z\left(\vec{B}\times\vec{C}\right)_x-\left(\vec{B}\times\vec{C}\right)_z a_x\right]+\hat{k}\left[a_x\left(\vec{B}\times\vec{C}\right)_y-\left(\vec{B}\times\vec{C}\right)_x a_y\right]$$

But, $\vec{B}\times\vec{C}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ b_x & b_y & b_z\\ c_x & c_y & c_z\end{vmatrix}=\hat{i}\left(b_y c_z-c_y b_z\right)+\hat{j}\left(b_z c_x-c_z b_x\right)+\hat{k}\left(b_x c_y-c_x b_y\right)$, so that

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right)=\hat{i}\left[a_y\left(b_x c_y-c_x b_y\right)-\left(b_z c_x-c_z b_x\right)a_z\right]+\hat{j}\left[a_z\left(b_y c_z-c_y b_z\right)-\left(b_x c_y-c_x b_y\right)a_x\right]$$

$$+\hat{k}\left[a_x\left(b_z c_x-c_z b_x\right)-\left(b_y c_z-c_y b_z\right)a_y\right]$$

$$=\hat{i}\left[b_x\left(a_y c_y+a_z c_z\right)-c_x\left(a_y b_y+a_z b_z\right)+b_x a_x c_x-c_x a_x b_x\right]$$

$$+\hat{j}\left[b_y\left(a_x c_x+a_z c_z\right)-c_y\left(a_x b_x+a_z b_z\right)+b_y a_y c_y-c_y a_y b_y\right]$$

$$+\hat{k}\left[b_z\left(a_x c_x+a_y c_y\right)-c_z\left(a_x b_x+a_y b_y\right)+b_z a_z c_z-c_z a_z b_z\right]$$

$$=\hat{i}\left[b_x\left(a_x c_x+a_y c_y+a_z c_z\right)-c_x\left(a_x b_x+a_y b_y+a_z b_z\right)\right]$$

$$+\hat{j}\left[b_y\left(a_x c_x+a_y c_y+a_z c_z\right)-c_y\left(a_x b_x+a_y b_y+a_z b_z\right)\right]$$

$$+\hat{k}\left[b_z\left(a_x c_x+a_y c_y+a_z c_z\right)-c_z\left(a_x b_x+a_y b_y+a_z b_z\right)\right]$$

$$=\hat{i}\left[b_x\left(\vec{A}\cdot\vec{C}\right)-c_x\left(\vec{A}\cdot\vec{B}\right)\right]+\hat{j}\left[b_y\left(\vec{A}\cdot\vec{C}\right)-c_y\left(\vec{A}\cdot\vec{B}\right)\right]+\hat{k}\left[b_z\left(\vec{A}\cdot\vec{C}\right)-c_z\left(\vec{A}\cdot\vec{B}\right)\right]$$

$$=\left(\vec{A}\cdot\vec{C}\right)\left[b_x\hat{i}+b_y\hat{j}+b_z\hat{k}\right]-\left(\vec{A}\cdot\vec{B}\right)\left[c_x\hat{i}+c_y\hat{j}+c_z\hat{k}\right]$$

So in summary,

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right)=\vec{B}\left(\vec{A}\cdot\vec{C}\right)-\vec{C}\left(\vec{A}\cdot\vec{B}\right).$$

Hence the name "Bac Cab". **Note**: that the first factors (B and C) in the two terms occur as vectors, but that the next two factors are part of a dot product and hence make a scalar factor for the vector they multiply.

To simplify $\left(\vec{A}\times\vec{B}\right)\times\vec{C}$ first use the anti-commutative property of the cross product twice and then apply the "Bac Cab" with the roles of C and A reversed.

$$\left(\vec{A}\times\vec{B}\right)\times\vec{C}=-\vec{C}\times\left(\vec{A}\times\vec{B}\right)=\vec{C}\times\left(\vec{B}\times\vec{A}\right)=\vec{B}\left(\vec{C}\cdot\vec{A}\right)-\vec{A}(\vec{C}\cdot\vec{B}).$$

So in summary,

$$\left(\vec{A}\times\vec{B}\right)\times\vec{C}=\vec{B}\left(\vec{A}\cdot\vec{C}\right)-\vec{A}\left(\vec{B}\cdot\vec{C}\right),$$

which could be called the "Bac Abc" theorem.

Note: $\left(\vec{A}\times\vec{B}\right)\times\vec{C}\neq\vec{A}\times\left(\vec{B}\times\vec{C}\right)$, in fact ,

$$\left(\vec{A}\times\vec{B}\right)\times\vec{C}-\vec{A}\times\left(\vec{B}\times\vec{C}\right)=\vec{B}\left(\vec{A}\cdot\vec{C}\right)-\vec{A}\left(\vec{B}\cdot\vec{C}\right)-\left[\vec{B}\left(\vec{A}\cdot\vec{C}\right)-\vec{C}\left(\vec{A}\cdot\vec{B}\right)\right]=\vec{C}\left(\vec{A}\cdot\vec{B}\right)-\vec{A}\left(\vec{B}\cdot\vec{C}\right)$$