Using the MacLaurin Series to Calculate Logarithms

The Maclaurin series for $\ln(1+x)$ is given by

$$\ln(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^{n} \qquad \textbf{(1)}.$$

This series converges to the logarithm for $-1<x\le 1$. This would appear of little use since I might very well need to calculate a $\ln(x)$ with $x > 2$. Nevertheless, this series can be used to calculate the natural logarithm of any positive number. Consider $q > 0$ then $\ln(q)$ is a real number. And we have the following identity.

$$\ln(q)=\ln\left(\frac{2q}{2}\right)=\ln\left(\frac{(q+1)-(1-q)}{(q+1)+(1-q)}\right)=\ln\left(\frac{1-(1-q)/(1+q)}{1+(1-q)/(1+q)}\right)=\ln\left(1-\frac{1-q}{1+q}\right)-\ln\left(1+\frac{1-q}{1+q}\right).$$

Now, $x=\dfrac{1-q}{1+q}$ has an absolute value less than 1 for any positive q. From equation (1),

$$\ln(q)=\ln(1-x)-\ln(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-x)^{n}}{n}-\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x)^{n}}{n}=-\sum_{n=1}^{\infty}\left[\frac{1-(-1)^{n+1}}{n}\right]x^{n}.$$

In this sum only the odd n terms are non zero. Writing n as $2j+1$, gives the following:

$$\ln(q)=-2\sum_{j=0}^{\infty}\frac{1}{2j+1}x^{2j+1}=-2\sum_{j=0}^{\infty}\frac{1}{2j+1}\left(\frac{1-q}{1+q}\right)^{2j+1}=2\sum_{j=0}^{\infty}\frac{1}{2j+1}\left(\frac{q-1}{q+1}\right)^{2j+1}.$$

This series converges to $\ln(q)$ for all positive q. For example, when $q=10$,

$\ln(q)=2\sum_{j=0}^{\infty}\frac{1}{2j+1}\left(\frac{9}{11}\right)^{2j+1}$. The sum $2\sum_{j=0}^{50}\frac{1}{2j+1}\left(\frac{9}{11}\right)^{2j+1}\approx 2.302585093$, which "is" the ln(10) to 9 decimal places.