The Limit Comparison Theorem

Given two sequences a_n and b_n if the limit $\lim_{n\to\infty}\left|\frac{a_n}{b_n}\right| = L$ exists and is a positive number then the infinite series $\sum_j^{\infty}|a_j|$ converges if and only if the infinite series $\sum_j^{\infty}|b_j|$ converges. This means that the two infinite series "do the same thing", i.e., they either both converge or they both diverge. It is impossible for one to converge while the other diverges. **Note:** The theorem does not assert that the two infinite series converge to the same value. They could be vastly different. Note also that this is a comparison that establishes absolute convergence.

Proof: Suppose that $\lim_{n\to\infty}\frac{a_n}{b_n} = L$ and $L > 0$. Then by the definition of the limit of a sequence for any positive ε there exists an M such that if $n > M$ then $\left|\left|\frac{a_n}{b_n}\right| - L\right| < \varepsilon$. This means that $-\varepsilon < \left|\frac{a_n}{b_n}\right| - L < \varepsilon$ and since ε can be any positive number, pick a value less than L. Thus, $0 < (L-\varepsilon) < \frac{|a_n|}{|b_n|} < (L+\varepsilon)$ or solving for $|a_n|$, $(L-\varepsilon)|b_n| < |a_n| < (L+\varepsilon)|b_n|$. Therefore, we have the inequality that $\sum_{j>M}^{N}|a_j| < (L+\varepsilon)\sum_{j>M}^{N}|b_j|$, so that if $\sum_j^{\infty}|b_j|$ converges, the increasing sequence of the partial sums of $\sum_{j>M}^{N}|a_j|$ are bounded above and hence must converge to a limit. Therefore the convergence of $\sum_j^{\infty}|b_j|$ implies convergence of $\sum_j^{\infty}|a_j|$.

Now, suppose that $\sum_j^{\infty}|a_j|$ converges. From the bottom half of $(L-\varepsilon)|b_n| < |a_n| < (L+\varepsilon)|b_n|$, since $L-\varepsilon>0$, $\sum_{j>M}^{N}|b_j| < \frac{1}{L-\varepsilon}\sum_{j>M}^{N}|a_j|$. Thus, the increasing sequence of the partial sums of $\sum_{j>M}^{N}|b_j|$ are bounded above and hence must converge to a limit. Therefore the convergence of $\sum_j^{\infty}|a_j|$ implies convergence of $\sum_j^{\infty}|b_j|$. Thus, the infinite series $\sum_j^{\infty}|a_j|$ converges if and only if the infinite series $\sum_j^{\infty}|b_j|$ converges. The statement in contra positive form is that if the limit $\lim_{n\to\infty}\left|\frac{a_n}{b_n}\right| = L$ exists and is a positive number then infinite series $\sum_j^{\infty}|a_j|$ diverges if and only if the infinite series $\sum_j^{\infty}|b_j|$ diverges.

The condition that L be positive was required so that $L-\varepsilon>0$. Suppose that L is either 0 or infinity. What can we say then? Not as much.

Sub case 1: Given two sequences a_n and b_n if the limit $\lim_{n\to\infty}\left|\frac{a_n}{b_n}\right| = 0$ and if the infinite series $\sum_j^{\infty}\left|b_j\right|$ converges, then the infinite series $\sum_j^{\infty}\left|a_j\right|$ also converges. In English this says that if the a_n decrease faster than the b_n , so that the ratio $\left|\frac{a_n}{b_n}\right|$ gets small and the b_n themselves decrease fast enough so that the infinite series $\sum_j^{\infty}\left|b_j\right|$ converges, then $\sum_j^{\infty}\left|a_j\right|$ must converge too.

Proof: In this case we only have the upper inequality. For any positive ε there exists an M such that if $n > M$ then $\left|\frac{a_n}{b_n}\right| < \varepsilon$. Solving for $\left|a_n\right|$, $\left|a_n\right| < \varepsilon\left|b_n\right|$. Therefore, $\sum_{j>M}^{N}\left|a_j\right| < \varepsilon\sum_{j>M}^{N}\left|b_j\right|$, so that if $\sum_j^{\infty}\left|b_j\right|$ converges, the increasing sequence of the partial sums of $\sum_{j>M}^{N}\left|a_j\right|$ are bounded above and hence must converge to a limit. Therefore the convergence of $\sum_j^{\infty}\left|b_j\right|$ implies the convergence of $\sum_j^{\infty}\left|a_j\right|$.

Sub case 2: Given two sequences a_n and b_n if the limit $\lim_{n\to\infty}\left|\frac{a_n}{b_n}\right| = \infty$ and if the infinite series $\sum_j^{\infty}\left|b_j\right|$ diverges, then the infinite series $\sum_j^{\infty}\left|a_j\right|$ also diverges. In English this says that if the a_n increase faster than the b_n , so that the ratio $\left|\frac{a_n}{b_n}\right|$ gets huge and the b_n themselves don't decrease fast enough to make the infinite series $\sum_j^{\infty}\left|b_j\right|$ converge, then $\sum_j^{\infty}\left|a_j\right|$ can't converge either.

Proof: For any positive U, there exists an M such that if $n > M$ then $\left|\frac{a_n}{b_n}\right| > U$. Solving for $\left|a_n\right|$, $\left|a_n\right| > U\left|b_n\right|$. Therefore, $\sum_{j>M}^{N}\left|a_j\right| > U\sum_{j>M}^{N}\left|b_j\right|$, so that if $\lim_{N\to\infty}\sum_{j>M}^{N}\left|b_j\right| = \infty$ the sequence of the partial sums of $\sum_{j>M}^{N}\left|a_j\right|$ must also increase without limit . Therefore the divergence of $\sum_j^{\infty}\left|b_j\right|$ implies the diveregence of $\sum_j^{\infty}\left|a_j\right|$.