 Calculus I : Project 1 /30

Name ____________________________ **Due 1/29/09**

1. Find the equation of the line, $\mathcal{L}$, that passes through $(-2, 3)$ and is parallel to the line $3x + 9y = 47$ (1 point)

The x intercept of $\mathcal{L}$ $=$ ______________

The y intercept of $\mathcal{L}$ $=$ ______________

The slope of $\mathcal{L}$ $=$ ______________

2. Given $f(x) = \sin(x)$ and $g(x) = \frac{1}{1-x^2}$. (1 point)
What is the domain of f ?

What is the range of f ?

What is the domain of g ?

What is the range of g ?

$f(x) \cdot g(x)$ $=$ ______________________________

$(g \circ f)(x)$ $=$ ______________________________

$(f \circ g)(x)$ $=$ ______________________________

3. In the table below indicate with a 'yes' or 'no' whether the graph of the stated equation has the stated symmetry. The abbreviation 'wrt' means 'with respect to' . (1 point)

Equation of Curve	**Symmetric wrt x-axis**	**Symmetric wrt y-axis**	**Symmetric wrt Origin**
$x^2 + 25 - y^2 = 0$			
$x^2 - 25 + y = 0$			
$(x-2)^2 - 16 - (y+1)^2 = 0$			
$x^3 - y^5 = x$			

4. Write an equation which generates each of the following curves. (1.5 points)

a) A circle of radius 4 centered at $(-1, 4)$.

b) A parabola with vertex at $(-2, 7)$, axis of symmetry $x = -2$, and passes through the point $(1, 25)$.

c) A parabola with vertex at $(5, 6)$, axis of symmetry $y = 6$, and passes through the point $(1, 2)$.

d) A cubic polynomial symmetric about the origin having a root at $x = -1$, and passing through the point $(2, 12)$.

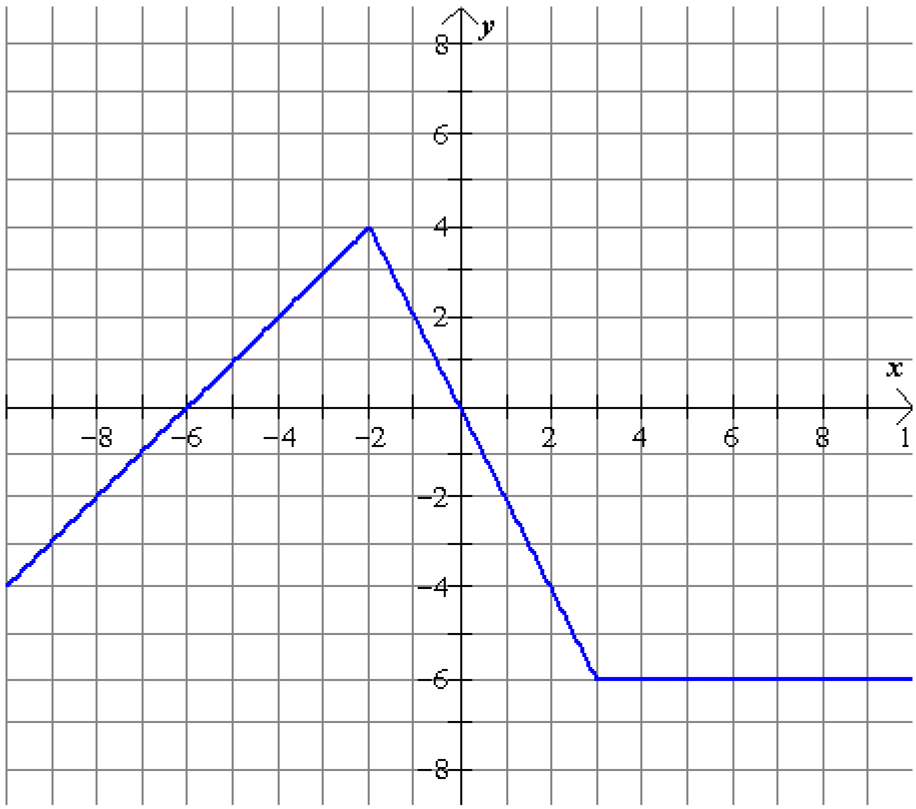
e) Identify which of the curves in parts a) to d) above define y as a function of x and which define x as a function of y

5. From the graph of the function, $f(x)$, shown below answer the following questions: (1.5 points)

a) For what intervals of the independent variable x is $f(x)$ strictly increasing?

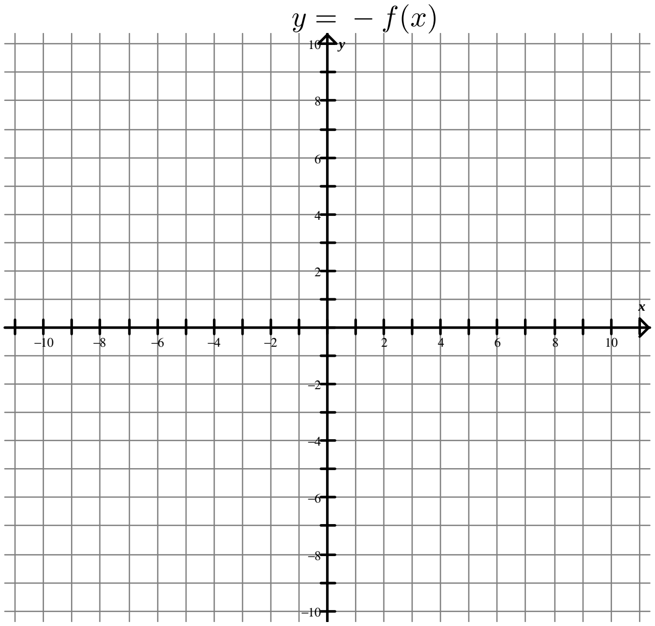
b) For what intervals of the independent variable x is $f(x)$ strictly decreasing?

c) Give an explicit (piecewise) definition of $f(x)$.

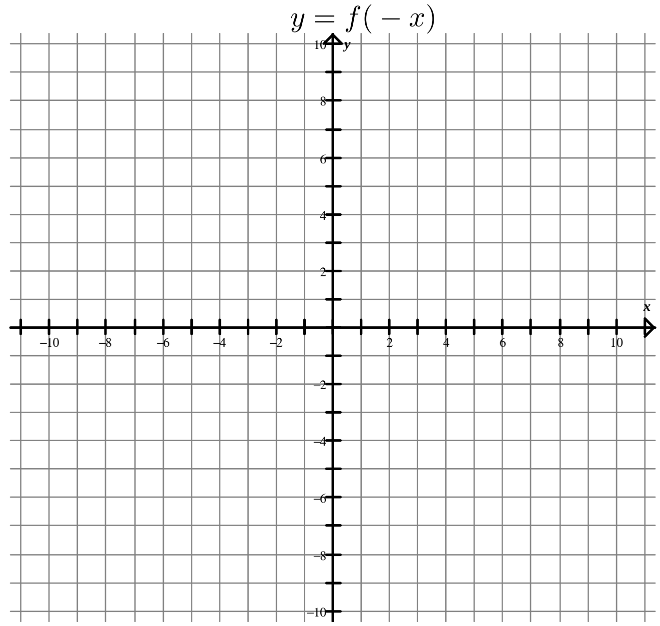


Sketch the following

d) $y = -f(x)$



e) $y = f(-x)$



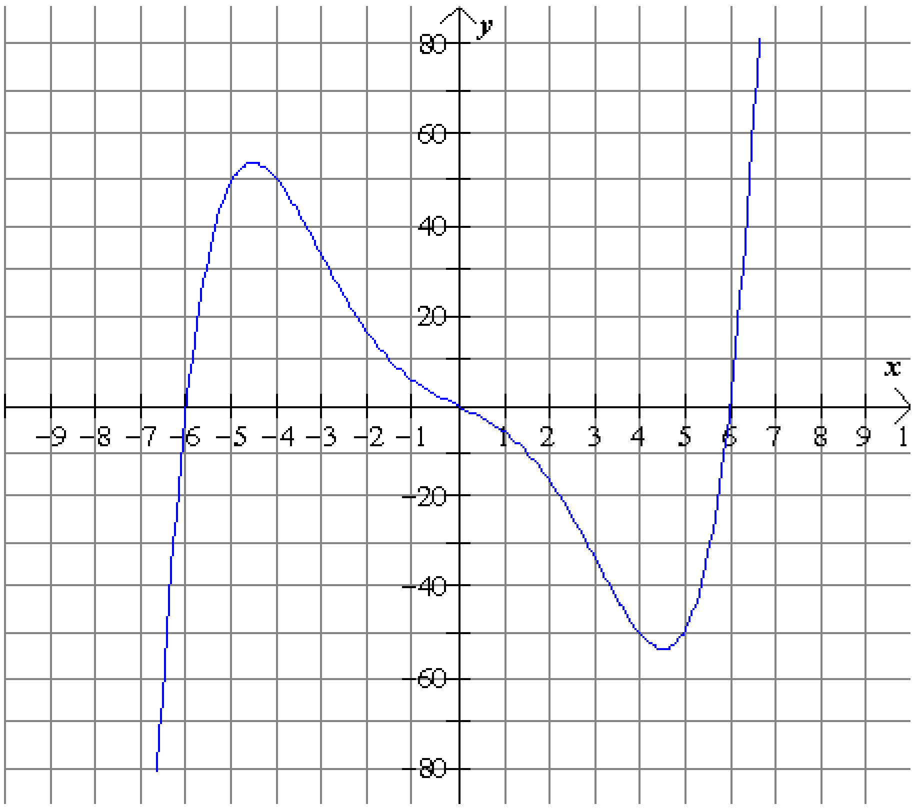
6. From the graph of the function, $f(x)$, shown below answer the following questions: (1 point)

a) What symmetry if any does $f(x)$ display?

b) For what intervals of the independent variable x is $f(x)$ increasing?

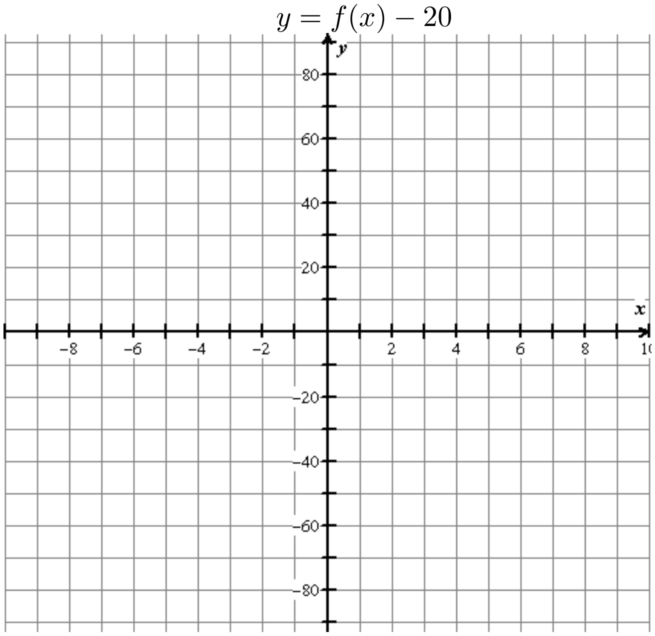
c) For what intervals of the independent variable x is $f(x)$ decreasing?

d) For what values of x does $f(x)$ have a relative or local minimum?

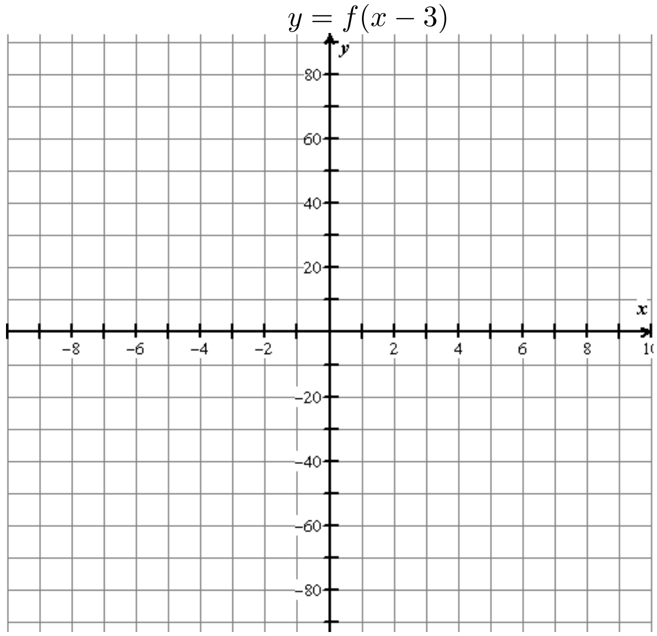


Sketch the following

e) $y = f(x) - 20$

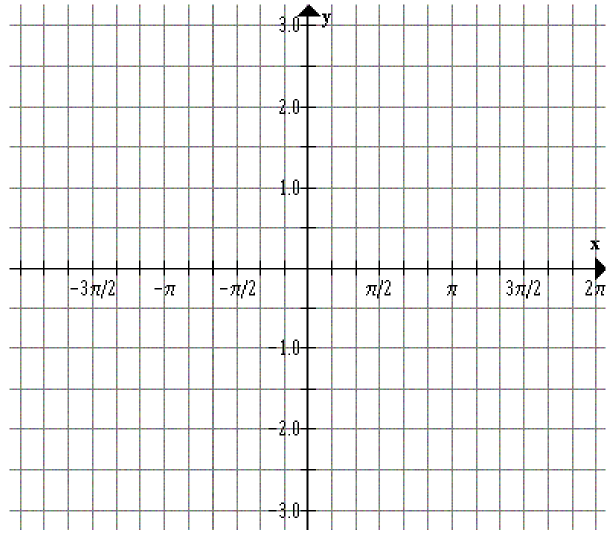


f) $y = f(x-3)$

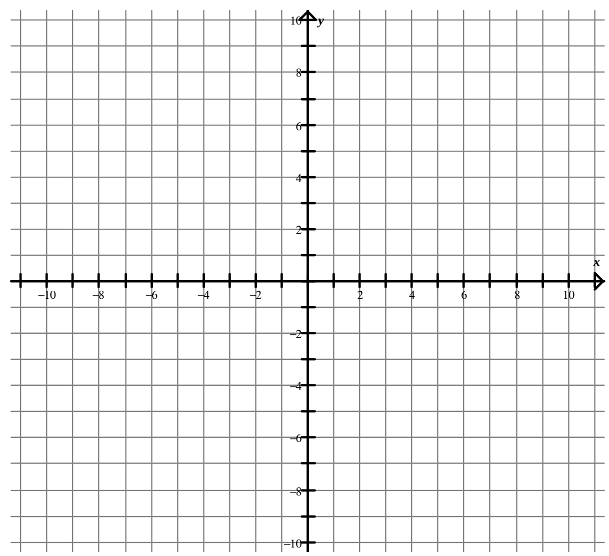


Sketch each of the following curves and label values on both axes. (1 point each)

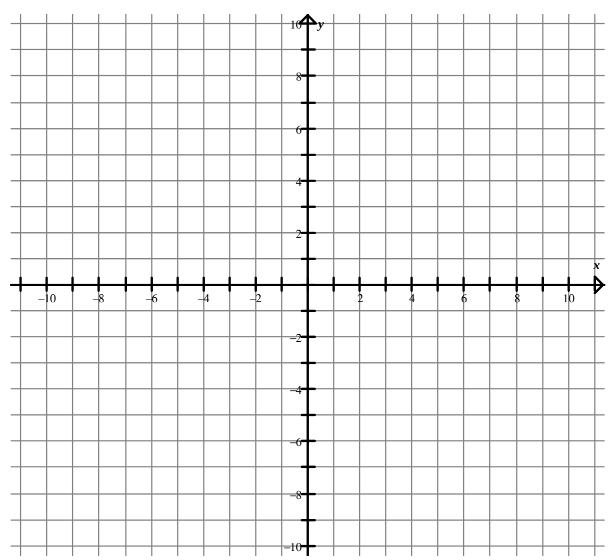
7. $y = 1 - 2\cos\left(2x - \pi\right)$



8. $2x^2 + 4x - y - 3 = 0$



9. $y^2 - 8y + 4x + 9 = 0$



10. Evaluate the following limits: (5 points)

Hint: Check your answers by graphing the appropriate function with a graphing calculator or Winplot.

a) $\lim_{x \to 2} 9x^2 - 36$ = ____________

b) $\lim_{x \to 2} \frac{9x^2-36}{x+2}$ = ____________

c) $\lim_{x \to 2} \frac{9x^2-36}{x-2}$ = ____________

d) $\lim_{t \to 0^-} \frac{t^2+2}{t}$ = ____________

e) $\lim_{t \to 0^+} \frac{t^2+2}{t}$ = ____________

f) $\lim_{x \to 0} \frac{x^2+5x}{x}$ = ____________

g) $\lim_{x \to \infty} \frac{7x + x\sin(3x)}{x^2}$ = ____________

h) $\lim_{x \to \infty} \frac{7x^2 + x\sin(3x)}{x^2}$ = ____________

i) $\lim_{x \to 0} \frac{(x-2)^3+8}{x}$ = ____________

j) $\lim_{x \to 3} x - \sqrt{x+6}$ = ____________

k) $\lim_{x \to 3} \frac{x-\sqrt{x+6}}{x-3}$ = ____________

l) $\lim_{x \to 6} \frac{\sqrt{x+30}-x}{x-6}$ = ____________

m) $\lim_{x \to 3} \frac{x^3-27}{x-3}$ = ____________

n) $\lim_{x \to \infty} x\sqrt{x^2+4} - x^2$ = ____________

o) $\lim_{x \to \infty} \sqrt{\frac{100x^8-93x^4-373x^2+419}{25x^8-85x^7+6x^6-12x^3+41x}}$ = ____________

11. (3.5 points)
In each case below conditions are stated about the limits of $f(x)$ and $g(x)$ as $x \to \infty$. For each case give an explicit example of functions $f(x)$ and $g(x)$ with these conditions or **show** why these conditions are **inconsistent**.

a) $\lim_{x \to \infty} f(x) = \infty$; $\lim_{x \to \infty} g(x) = \infty$; $\lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty$

b) $\lim_{x \to \infty} f(x) = \infty$; $\lim_{x \to \infty} g(x) = \infty$; $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$

c) $\lim_{x \to \infty} f(x) = \infty$; $\lim_{x \to \infty} g(x) = \infty$; $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 2$

d) $\lim_{x \to \infty} f(x) = \infty$; $\lim_{x \to \infty} g(x) = \infty$; $\lim_{x \to \infty} f(x) - g(x) = 9$

e) $\lim_{x \to \infty} f(x) = \infty$; $\lim_{x \to \infty} g(x) = \infty$; $\lim_{x \to \infty} f(x) - g(x) = -\infty$

f) $\lim_{x \to \infty} f(x) = \infty$; $\lim_{x \to \infty} g(x) = \infty$; $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 3$; $\lim_{x \to \infty} f(x) - g(x) = \infty$

g) $\lim_{x \to \infty} f(x) = \infty$; $\lim_{x \to \infty} g(x) = \infty$; $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 4$; $\lim_{x \to \infty} f(x) - g(x) = 8$

12. (2points) The $\lim_{x \to a} f(x) = L$ if and only if for every ϵ, $\epsilon > 0$, there exists a δ, $\delta > 0$, such that whenever $0 < |x - a| < \delta$, $|f(x) - L| < \epsilon$.

a) For $f(x) = 7 - 2x$ determine $L = \lim_{x \to a} f(x)$ and an expression for δ involving ϵ that insures that $|f(x) - L| < \epsilon$.

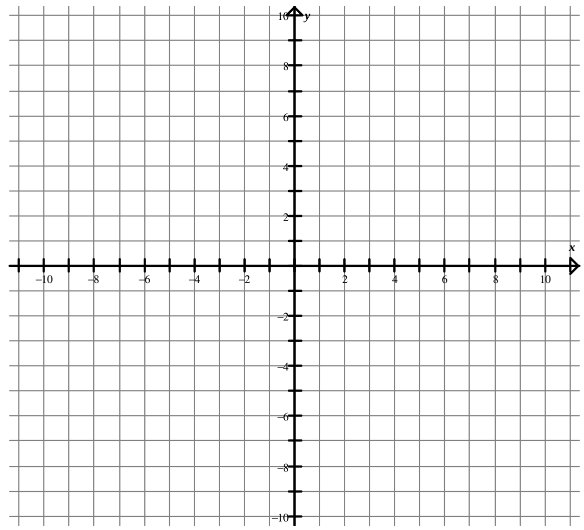
b) For $f(x) = \frac{4x^2 - a^2}{2x - a}$ determine $L = \lim_{x \to \frac{a}{2}} f(x)$ and an expression for δ involving ϵ that insures that $|f(x) - L| < \epsilon$.

c) The $\lim_{x \to \infty} f(x) = L$ if and only if for every ϵ, $\epsilon > 0$, there exists a M, such that whenever $x > M$, $|f(x) - L| < \epsilon$.
For $f(x) = \frac{6x}{\sqrt{x^2 + 4x}}$ determine $L = \lim_{x \to \infty} f(x)$ and an expression for M involving ϵ that insures that $|f(x) - L| < \epsilon$.

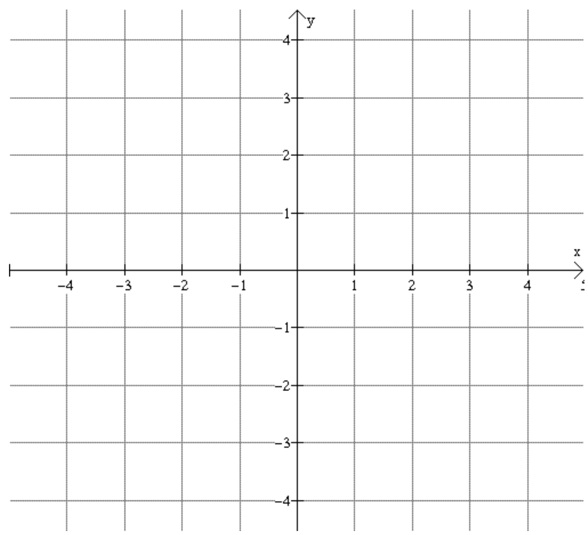
d) The $\lim_{x \to a^-} f(x) = \infty$ if and only if for every M, $M > 0$, there exists a δ, $\delta > 0$, such that whenever $a - \delta < x < a$, $f(x) > M$. For positive a the $\lim_{x \to a^-} \frac{b^2}{\sqrt{a^2 - x^2}} = \infty$. Determine an expression for δ involving M that insures that $f(x) > M$.

Graph each of the following functions. State where in its domain each function is continuous. (1 point each)

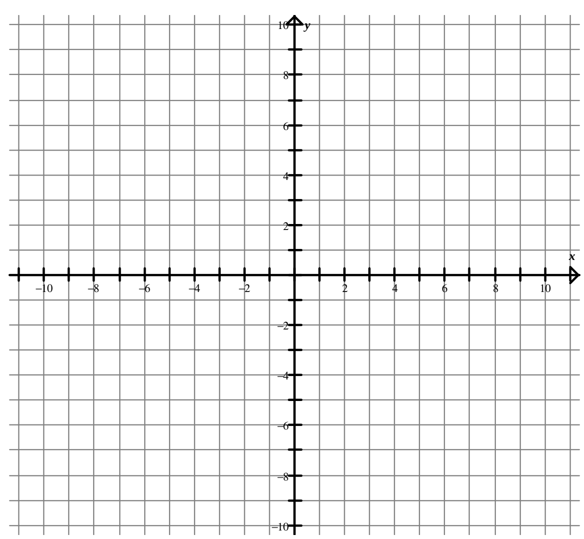
13. $f(x) = \begin{cases} 7 - x & \text{if } x < 1 \\ 3(x+1) & \text{if } x \geq 1 \end{cases}$



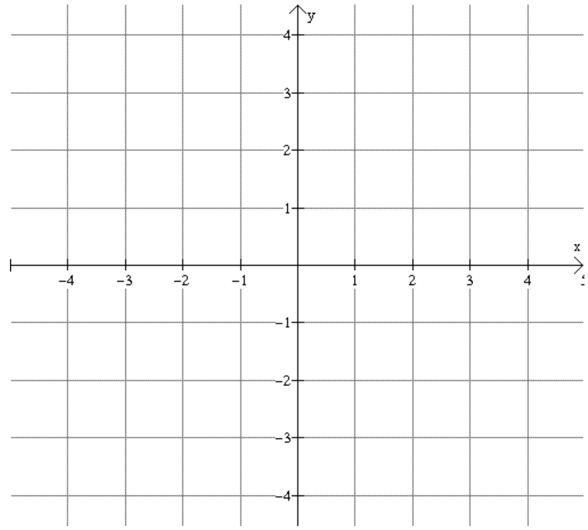
14. $f(x) = |x^2 - 1|$



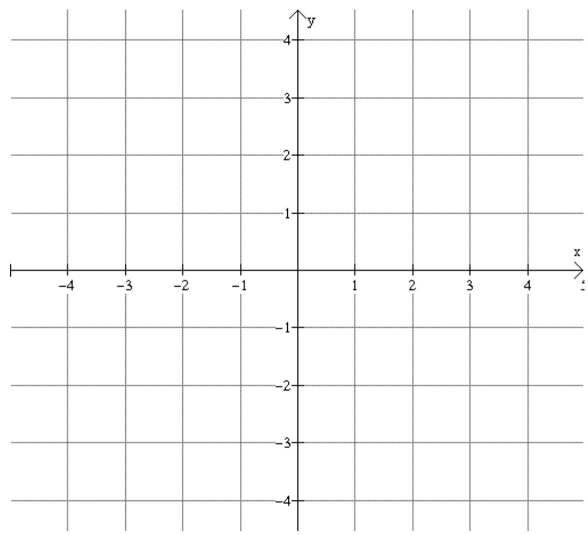
15. $f(x) = \begin{cases} |x| - 2 & \text{if } x \leq -2 \\ x - 2 & \text{if } x > -2 \end{cases}$



16. $f(x) = \begin{cases} x^3 & \text{if } |x| \le 1 \\ x^2 & \text{if } |x| > 1 \end{cases}$



17. $f(x) = \begin{cases} x^2 & \text{if } |x| \le 1 \\ |x| & \text{if } |x| > 1 \end{cases}$



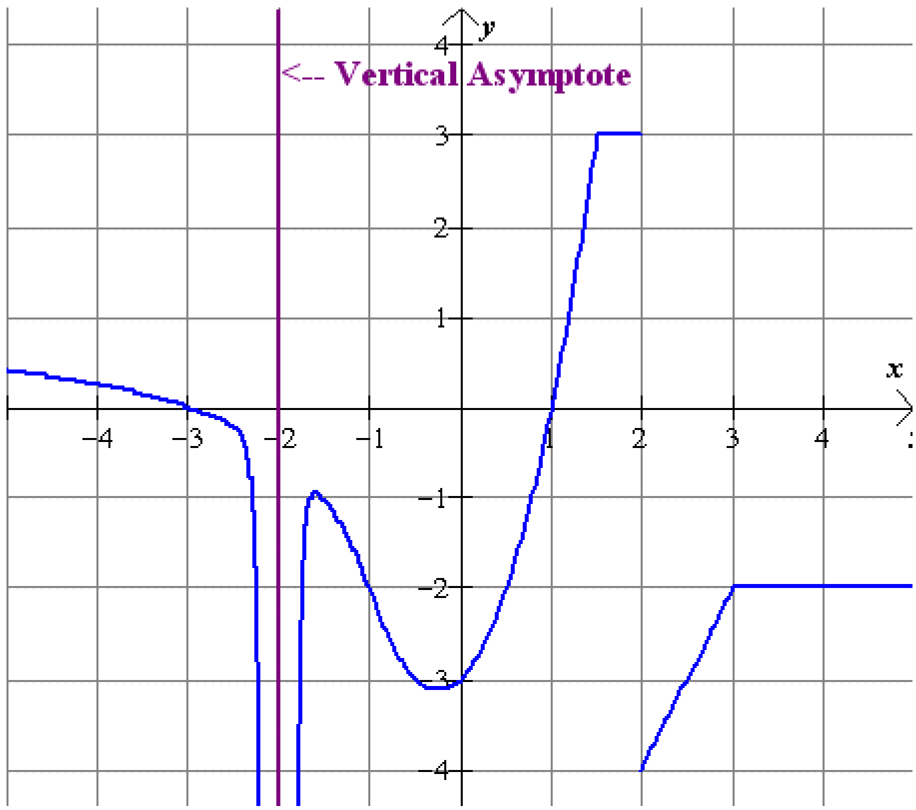
18. Which function below has a continuous extension and state the formula for this extension. (1 point)

$f(x) = \frac{x^2+4}{x-2}$ $g(x) = \frac{x^3-8}{x-2}$

19. Determine the value of a that makes the following function continuous: (1 point)

$$f(x) = \begin{cases} |x-2a| & \text{if } x < -3 \\ |a| & \text{if } x = -3 \\ 2x+7 & \text{if } x > -3 \end{cases}$$

20. Below is a plot of the function $f(x)$ defined on $(-5, 5)$. (2.5 points)



Determine the following:

a) The roots (zeros) of $f(x)$.

b) The values in $(-4, 4)$ where $f(x)$ is discontinuous.

c) $\lim_{x \to 0} f(x)$ = ______________

d) $\lim_{x \to 2^+} f(x)$ = ______________

e) $\lim_{x \to 2^-} f(x)$ = ______________

f) $\lim_{x \to -3} x^3 f(x)$ = ______________

g) $\lim_{x \to 2^-} f(x+1)f(x)$ = ______________

h) $\lim_{s \to 0} \frac{f(2.5+s)-f(2.5)}{s}$ = ______________

Self-Assessment: (2 bonus points)

a) Describe three strengths in your perfomance on this project. Include why each is a strength.

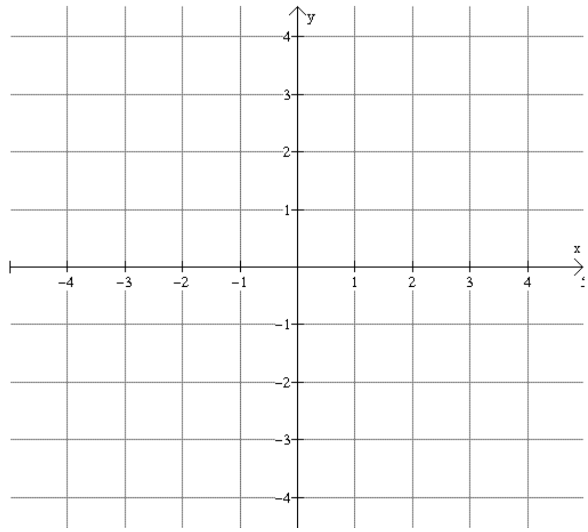
b) What are three things that could be improved about your performance on this project. Explain specifically how you will make these improvements.

c) Identify two things about this project which are still unclear to you.

d) Identify two insights that you have acquired in doing this project.

Name ______________________________ **Due 2/16/09**

1. Let $f(x) = |4 - x^2|$ (2 points)
a) Sketch $f(x)$



b) Is f a continuous function? Explain your answer.

c) What is $f'(x)$ for $x > 2$?

d) What is $f'(x)$ for $|x| < 2$?

e) What is $f'(x)$ for $x < -2$?

f) Explain whether or not $f'(x)$ exists at $x = -2$
.

2. (4 points) The 'standard' numerical approximation to the derivative (i.e., the slope of the tangent line) of a function f at a point x is the value of the difference quotient

Standard Approx $= \frac{f(x+\Delta x)-f(x)}{\Delta x}$, where Δx is a 'small' but non-zero number. The 'central difference' numerical approximation to the derivative of a function *f* at a point x is the value of the quotient

Central Difference Approx $= \frac{f(x+\Delta x)-f(x-\Delta x)}{2\Delta x}$,where Δx is a 'small' but non-zero number.

Given the following functions : $f(x) = 3x^2 + 1$; $g(x) = \sqrt{4x+1}$; $h(x) = \frac{1}{\sqrt{4x+1}}$

With a calculator, determine both numerical approximations of the derivatives for $f(x)$ at $x = -1, 0$ and 2, for $g(x)$ at $x = 6$, and for $h(x)$ at $x = 6$ and $x = 12$. Use the three values of $0.1, 0.05$ and 0.01 for Δx . In addition, for each function and given value of x , determine the exact value of the derivative **by its definition in terms of a limit**. Fill in the tables below giving **all answers as decimals** to at least four decimal places.

Function	x	**Standard Approximation of the Derivative**			**Exact Derivative**
		$\Delta x = 0.1$	$\Delta x = 0.05$	$\Delta x = 0.01$	
f	-1				
f	0				
f	2				
g	6				
h	6				
h	12				

Function	x	**Central Difference Approximation of the Derivative**			**Exact Derivative**
		$\Delta x = 0.1$	$\Delta x = 0.05$	$\Delta x = 0.01$	
f	-1				
f	0				
f	2				
g	6				
h	6				
h	12				

For both forms of approximation, how well do the approximate and exact results agree ?

Which form of approximation gives the best answers?

For both forms of approximation, what happens as Δx gets smaller ?

Explain whether or not this should happen ?

3. For each statement below indicate the dependent and independent variables, then interpret what the statement is saying about the derivative(s) of the dependent variable with respect to the independent variable. (2 points)
a) As the production of an appliance increases its price falls.

b) The inflation rate is increasing.

c) The temperature is still climbing, but not as fast as an hour ago.

d) The car gradually slows to a complete stop.

4. Evaluate the following limits: (2 points)

a) $\lim_{x \to 0} \frac{8x - \sin(3x)}{x} =$ ____________________

b) $\lim_{x \to 0} x^5 \sin\left(\frac{\alpha - \beta}{x^5}\right) =$ ____________________

c) $\lim_{x \to \infty} x^5 \sin\left(\frac{\alpha - \beta}{x^5}\right) =$ ____________________

d) $\lim_{x \to 0} \cos\left(\frac{\pi \sin(2x^2)}{x \sin(x)}\right) =$ ____________________

5. (8 points) Compute the following derivatives for the given functions. Compute the following derivatives.
The QuickMath website, which uses Mathematica to calculate derivatives, is at
www.hostsrv.com/webmab/app1/MSP/quickmath/02/pageGenerate?site=quickmath&s1=calculus&s2=differentiate&s3=basic .
It can be used to easily verify your results.

$$f(x) = \left(4x^2 + 6x - 3\right)^6$$

a) $f'(x) =$ ____________________

b) $f''(x) =$ ____________________

$f(x) = \left(4x^2 + 6x - 3\right)^{-6}$

c) $f'(x) \quad = $ ______________________________

$f(x) = 5z^2 - 8z - 3$, where $z = \sin\left(\frac{\pi x}{2}\right)$

d) $f'(x) \quad = $ ______________________________

$f(x) = \frac{14x^7 - 5x^3 - 4x - 7}{x^4}$

e) $f'(x) \quad = $ ______________________________

f) $f''(x) \quad = $ ______________________________

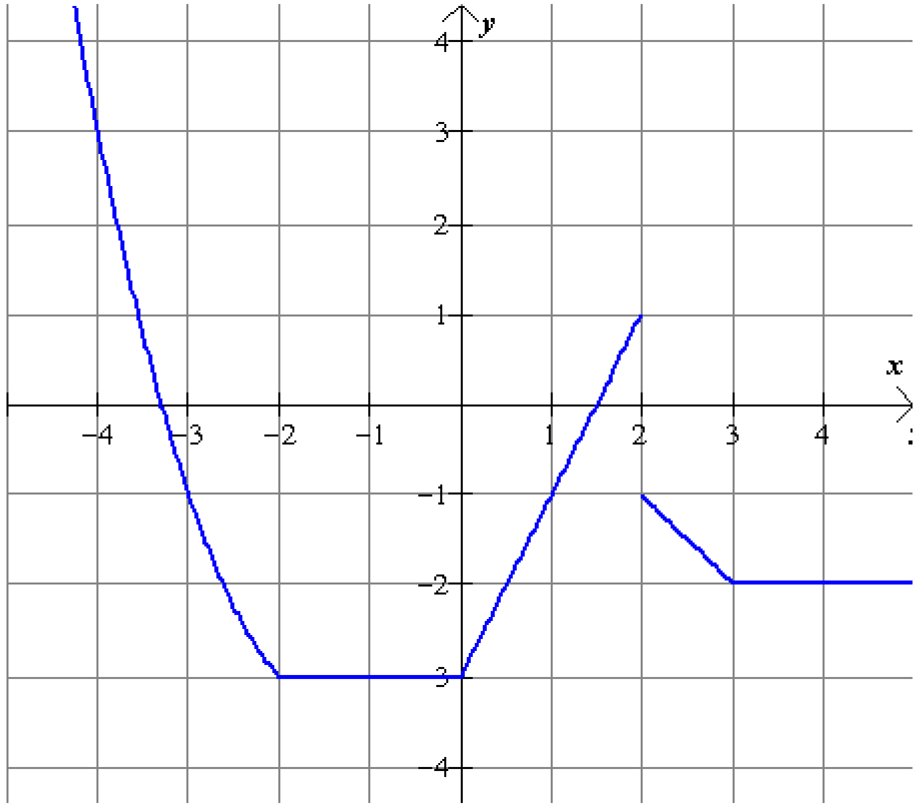
$f(x) = x^2 \sin\left(-\frac{3}{x}\right)$

g) $f'(x) \quad = $ ______________________________

$f(\theta) = \tan\left(\frac{\sin(4\theta)}{\theta}\right)$

h) $f'(\theta) \quad = $ ______________________________

6. Below is a plot of the function $f(x)$ defined on $(-5, 5)$. (3 points)



Evaluate the following:

a) $f'\left(-\frac{3}{2}\right)$ = ______________

b) $f'(1)$ = ______________

c) $f'(4)$ = ______________

d) $\lim_{x \to -2} f(x)$ = ______________

e) $\lim_{x \to 2^+} f(x)$ = ______________

f) $\lim_{x \to 2^-} f(x)$ = ______________

g) $\lim_{x \to 1^+} f(x+1)f(x)$ = ______________

h) $\lim_{x \to 1^-} f(x+1)f(x)$ = ______________

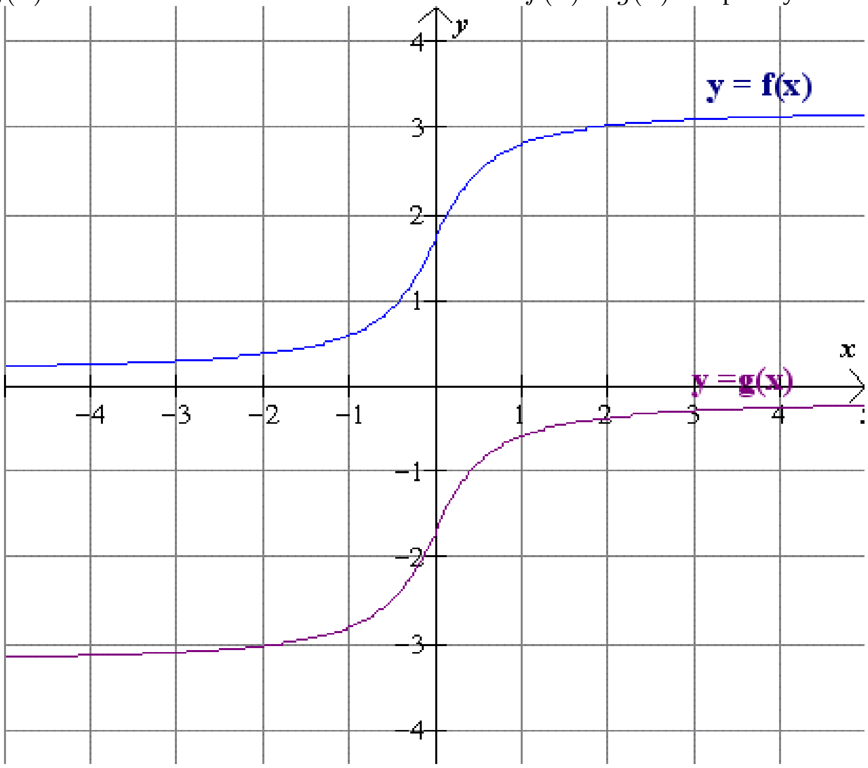
i) $\lim_{s \to 0^-} \frac{f(s)-f(0)}{s}$ = ______________

j) $\lim_{s \to 0^+} \frac{f(s)-f(0)}{s}$ = ______________

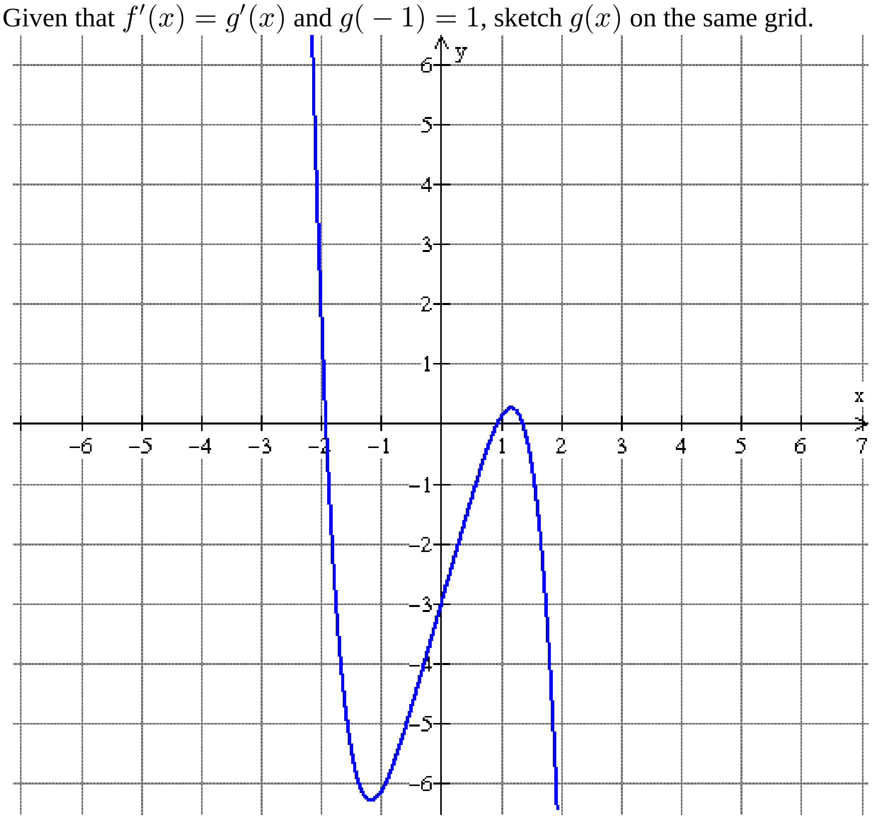
k) At what values in its domain is $f(x)$ discontinuous?

l) At what values in its domain where $f(x)$ is continuous does $f'(x)$ fail to exist?

7. (1 point)
a) The graphs of two functions $f(x)$ and $g(x)$ are shown below. What is the derivative of $f(x)-g(x)$? Explain your answer.



b) A graph of $y=f(x)$ is shown below. Given that $f'(x)=g'(x)$ and $g(-1)=1$, sketch $g(x)$ on the same grid.



8. (1 point) Given $f(s) = \frac{s^2+1}{s^2-4s}$

a) Evaluate $\lim_{s \to \infty} f(s)$ $=$ ______________________________

b) Evaluate $f'(s)$ $=$ ______________________________

c) Evaluate $\lim_{s \to \infty} f'(s)$ $=$ ______________________________

d) Explain whether your answer to part a) is consistent with your answer to part c) .

9. (2 points)
a) Find the slope of the curve defined by $x^2 + y^2 + 1 = 4x - 6y + 2xy$ at the point $(2\,,\,-3\,)$.

b) Find the equation of the tangent line to the above curve at $(2\,,\,-3\,)$.

c) Find the equation of the normal to the above curve at $(2\,,\,-3\,)$.

d) Find $\frac{d^2y}{dx^2}$ at $(2\,,\,-3\,)$.

10. (2 points)
The displacement s from the origin of a non-uniformly accelerated particle is given by the following function of time t :
sec is time unit of second and m is a distance unit of m.

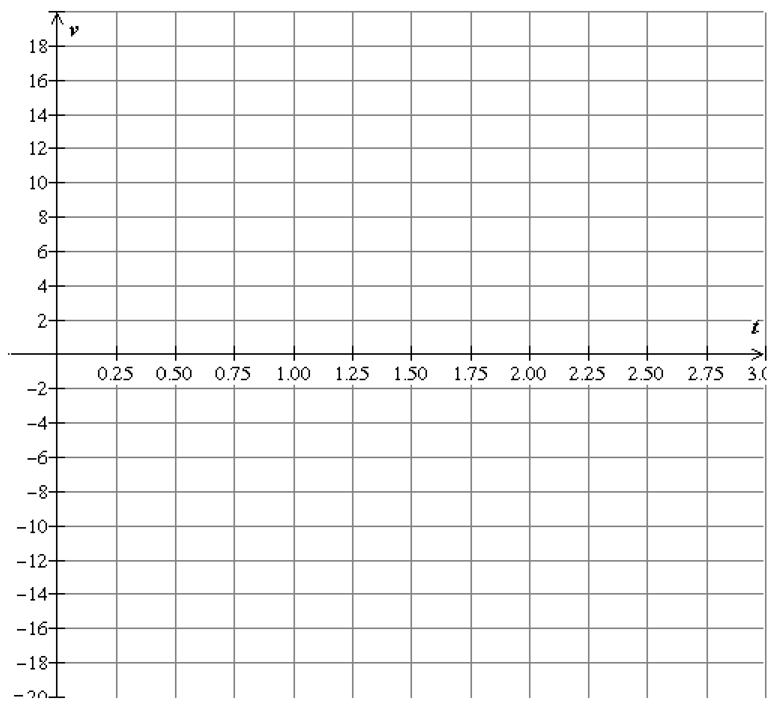
$$s(t) = \frac{1\text{m}}{3\text{sec}^3}t^3 + \frac{2\text{m}}{\text{sec}^2}t^2 - \frac{5\text{m}}{\text{sec}}t \quad ; \text{ for } 0 \text{ sec } \leq t \leq 3 \text{ sec}$$

a) Determine the velocity, v , as a function of time. v = ____________________

b) Determine the acceleration, a , as a function of time. a = ____________________

c) Sketch the velocity function from $t = 0$ sec to $t = 3$ sec .

Velocity in m/sec versus time in sec



d) When is the particle moving to the left? When is it moving to the right?

e) What is the net displacement, Δs , from $t = 0$ sec to $t = 3$ sec ? Δs = ____________________

f) What is the average velocity over the time interval $t = 0$ sec to $t = 3$ sec ? $\overline{v}$ = ____________________

g) What is the particle's initial velocity?

h) When does the particle stop moving?

i) When is the particle moving the fastest?

Do one of the following two problems. You may bo **both** for three **bonus** points!

11. (3 points)

Let $\mathcal{L}$ be the graph of $y = x^3$ and $\mathcal{M}$ be the graph of $y = Ax^3 + Bx^2 + Cx + D$ for real constants $A,\ B,\ C,$ and D with $A \neq 0$.

a) If $a \neq 0$ then the tangent line to $\mathcal{L}$ at $x = a$ intersects $\mathcal{L}$ at a second point with $x = b, b \neq a$. Determine a formula for b in terms of a .

b) Calculate the ratio of the slope of $\mathcal{L}$ at $x = b$ to the slope of $\mathcal{L}$ at $x = a$.

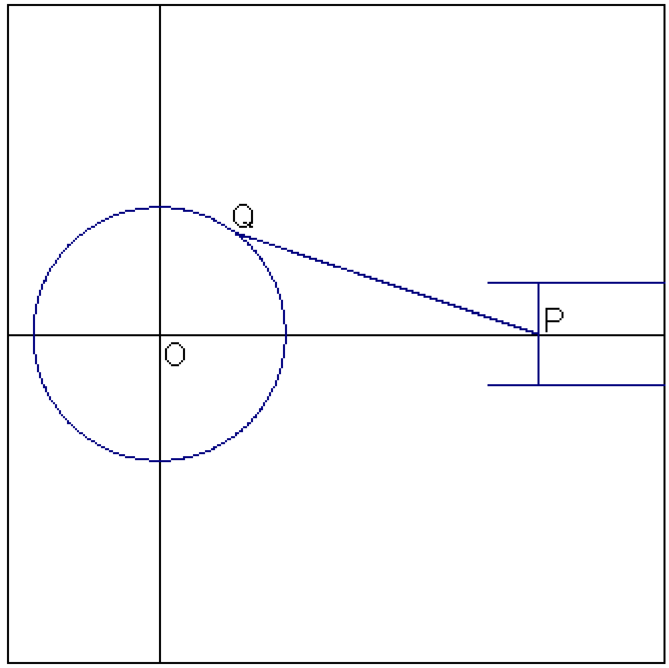
c) Can any line be tangent to $\mathcal{L}$ at more than one point? Explain your reasoning.

d) If the tangent line to $\mathcal{M}$ at $x = a$ is parallel to the tangent line to $\mathcal{M}$ at $x = b$, determine a formula for b in terms of a .

e) Can any line be tangent to $\mathcal{M}$ at more than one point? Explain your reasoning.

f) What is the minimum of a polynomial of degree greater than one so that a line is tangent to its graph at more than one point. Explain your reasoning.

12. (3 points) In a reciprocating internal combustion engine each piston **P** is housed in a cylinder and attached to the rim of a rotating crankshaft by connecting rod **QP** . The combustion of the fuel causes the piston to move back and forth in its cylinder, and, in response, the crankshaft rotates. Among factors which determine engine stress and wear are the velocity and acceleration of the pistons. **Warning : Messy derivatives ahead! Try a CAS** .



Assume a circular crankshaft of radius r and a connecting rod of length L ($L =$ length of **QP**). If the crankshaft is rotating at an angular velocity of ω (measured in radians per second, $\frac{2\pi}{\omega}$ is the time for one complete revolution of the crankshaft), answer the following questions.

a) If the origin in time, t, is taken to be when the piston is at maximum compression (when **Q** is on the positive x axis), obtain a formula for the distance of the piston **P** from the center of the crankshaft at any time t . You may assume, but it's actually irrelevant, that the crankshaft rotates counterclockwise. Your formula will be a function of t , ω , L and r .

Distance from **P** to center of crankshaft $=$ ____

b) Obtain a formula for the velocity of the the piston.

Velocity of the piston $=$ ____

c) Obtain a formula for the acceleration of the the piston.

Acceleration of the piston $=$ ____

d) Is the motion of the piston actually sinusoidal [i.e., of the form $A\sin(at+\delta)+B$]? Explain your answer.

Assume that the crankshaft radius is 5.0 cm, the connecting rod is 12.0 cm and the crankshaft rotates at 2400 rpm.
(1 rpm $=1$ revolution per minute)
e) Using either a graphing calculator or a computer program estimate the maximum velocity of the crankshaft to the nearest meter per second.

Maximum velocity of the piston $=$ ______________________________ $\frac{\text{m}}{\text{s}}$

f) Using either a graphing calculator or a computer program estimate the maximum acceleration of the crankshaft to the nearest 10 meters per second squared.

Maximum acceleration of the piston $=$ ______________________________ $\frac{\text{m}}{\text{s}^2}$

Self-Assessment: (2 bonus points)
a) Describe three strengths in your perfomance on this project. Include why each is a strength.

b) What are three things that could be improved about your performance on this project. Explain specifically how you will make these improvements.

c) Identify two things about this project which are still unclear to you.

d) Identify two insights that you have acquired in doing this project.

 Calculus I : Project 3 /30

Name ______________________________ **Due 3/03/09**

1. Given $f(x) = \frac{1}{3x-2}$ fill in the following: (2.5 points)

a) $f^{-1}(x)$ = ________________________________ b) What is the domain of f ?

c) $f^{-1}\left(f(x)\right)$ = ________________________________ d) What is the range of f?

e) $f\left(f^{-1}(x)\right)$ = ________________________________ f) What is the domain of f^{-1} ?

g) $\frac{d}{dx}\left(f^{-1}(x)\right)$ = ________________________________ h) What is the range of f^{-1} ?

i) Use **WinPlot** generate and attach the computer plots of the following five curves on a single graph.
Use the window : left $=$ down $=$ -5 ; right $=$ up $=$ 5 ;

	WinPlot Equa format		formula to enter
Curve 1.	1. **Explicit**	low x = -5 hi x =5	f(x) = $\boxed{f(x)}$
Curve 2.	1. **Explicit**	low x = -5 hi x =5	f(x) = $\boxed{x}$
Curve 3.	2. **Parametric**	low t = -5 hi t = 5	f(t) = $\boxed{f(t)}$; g(t) = $\boxed{t}$
Curve 4.	1. **Explicit**	low x = -5 hi x =5	f(x) = $\boxed{f^{-1}(x)}$
Curve 5.	2. **Parametric**	low t = -5 hi t = 5	f(t) = $\boxed{f^{-1}(t)}$; g(t) = $\boxed{t}$

j) What is true about the line $y = x$, with respect to the Curve 1 and Curve 3?

k) How do the Curve 1 and Curve 5 compare? Explain why this is not surprising.

l) How do the Curve 3 and Curve 4 compare? Explain why this is not surprising.

2. Evaluate and/or simplify the following: (4.5 points)

a) $\sin\left(\tan^{-1}\left(-\frac{1}{3}\right)\right)$ = ________________

b) $\tan\left(\sin^{-1}\left(-\frac{1}{3}\right)\right)$ = ________________

c) $\tan\left(\cos^{-1}\left(-\frac{1}{3}\right)\right)$ = ________________

d) $\sin\left(\cos^{-1}\left(-\frac{1}{3}\right)\right)$ = ________________

e) $-\pi^{-1}\left(\tan^{-1}(-1) - \tan^{-1}(3^{-\frac{1}{2}})\right)$ = ________________

f) $\lim_{x \to 0} \cos^{-1}\left(\sin^{-1}(4x)\right)$ = ________________

g) $\ln(e^{\sin^{-1}(x)})$ = ________________

h) $e^{-4\ln|x|}$ = ________________

i) $\dfrac{\log_5(x^7)}{\ln x}$ = ________________

3. Evaluate the following derivatives: (6.5 points)

a) $f(x) = \tan^{-1}(e^{-\frac{x}{3}})$ $f'(x) =$ ________________________________

b) $f(x) = \tanh(4x)$ $f'(x) =$ ________________________________

c) $f(x) = e^{-\sin^{-1}(x)} = \exp\Big(- \sin^{-1}(x)\Big)$ $f'(x) =$ ______________________

d) $f(x) = x^{-2x}$; for $x > 0$ $f'(x) =$ ______________________

e) $f(x) = \sqrt[8]{\frac{xe^{16x}}{x^2+1}}$ $f'(x) =$ ______________________

f) $f(x) = \log_8(e^x)$ $f'(x) =$ ______________________

g) $f(x) = 6^{\ln x^2}$ $f'(x) =$ ______________________

h) $f(x) = \tan(\tan^{-1} x)$ $f'(x) =$ ______________________

i) $f(x) = \cos\Big(\sec^{-1}(e^x)\Big)$ $f'(x) =$ ______________________

j) $f(x) = e^{\sin(x)}$ $f'(x) =$ ______________________

k) $f(x) = e^{\ln(2x)}$ $f'(x) =$ ______________________

l) $f(x) = \ln(\sqrt{x^2 + x})$ $f'(x) =$ ______________________

m) $f(x) = \left[\ln(x^3)\right]^3$ $f'(x) =$ ______________________

4. (2 points)

a) Given $y > x$ and $f(s) = s^2 + 4s - 5$, find t between x and y such that $f'(t) = \frac{f(y) - f(x)}{y - x}$.

b) For real numbers r and z , can $|\cos(r) - \cos(z)|$ ever be bigger than $|r - z|$? Justify your answer.

5. (2 points) The volume of a sphere is given by the formula : $V = \frac{4\pi}{3} r^3$, where V is the volume and r is the radius.

a) What is the differential of the volume of a sphere in terms of r and dr ?

b) If the radius of a sphere increases by 0.08% , what is the approximate % increase in the sphere's volume?

6. (2.5 points)
Find both the tangent (linear approximation) approximation $L(x)$ and the quadratic approximation $P(x)$ to the function $f(x) = \sqrt{6x+1}$ about $x = 4$.

a) $L(x)$ = ____________________

b) $P(x)$ = ____________________

Using a calculator give the following to at least four decimal places.

c) $L(4.025)$ = ____________________

d) $P(4.025)$ = ____________________

e) $f(4.025)$ = ____________________

f) What is the percent error of the linear approximation at $x = 4.025$?

g) What is the percent error of the quadratic approximation at $x = 4.025$?

7. Evaluate the following limits: (6 points)

a) $\lim_{x \to 2} \frac{5x^3-17x-6}{x^2-4}$ = ____________________

b) $\lim_{x \to 2} \frac{5x^3-17x-6}{x^2+4}$ = ____________________

c) $\lim_{x \to \infty} \frac{x}{\sqrt{9x^2+49x-119}}$ = ____________________

d) $\lim_{x \to \pi} \frac{\pi^2-x^2}{\sin(x)}$ = ____________________

e) $\lim_{x \to -\pi} \frac{\tan(x)}{x+\pi}$ = ____________________

f) $\lim_{x \to \frac{\pi}{2}} \left(x - \frac{\pi}{2}\right)\tan(x)$ = ____________________

g) $\lim_{x \to 0} \frac{\sinh(\alpha x)}{x}$ = ____________________

h) $\lim_{x \to \infty} x - \sqrt{x^2 - 4x - 26}$ = ____________________

i) $\lim_{x \to \infty} x \ln\left(1 + \frac{\alpha}{x}\right)$ $=$ ____________________

j) $\lim_{x \to \infty} e^{2x} \left(1 - \tanh(x)\right)$ $=$ ____________________

k) $\lim_{x \to \infty} \frac{\left(1-\frac{\alpha}{x}\right)^x}{\left(1+\frac{\alpha}{x}\right)^x}$ $=$ ____________________

l) $\lim_{x \to \infty} e^{-2x}(x^{14} + 11x^{10} - 12x^8 + 19x^4)$ $=$ ____________________

8. (3 points)
The function $f(x) = 2x^3 + 17x^2 + 19x - 3$ has three real roots or zeroes.
a) Use synthetic division or a calculator to determine intervals which contain each of the three roots. Use each interval to obtain an estimate of the root of $f(x)$ on that interval . Call this estimate x_0 .

b) What is the formula for the $n + 1'$st estimate of the positive root of $f(x)$ in terms of the n'th estimate using Newton's method?

c) Fill in the table below for the successive Newton's estimates for all three real roots of $f(x)$. Iterate until each root is obtained to at least the nearest hundred-thousandth.

	Root 1		**Root 2**		**Root 3**	
$\boldsymbol{n}$	$\boldsymbol{x_n}$	$\boldsymbol{f(x_n)}$	$\boldsymbol{x_n}$	$\boldsymbol{f(x_n)}$	$\boldsymbol{x_n}$	$\boldsymbol{f(x_n)}$
0						
1						
2						
3						
4						
5						
6						
7						
8						

9. (1 point) Below are shown the graphs of $f(x)$, $f'(x)$, and $f''(x)$. Identify which curve is which function and explain your reasoning.

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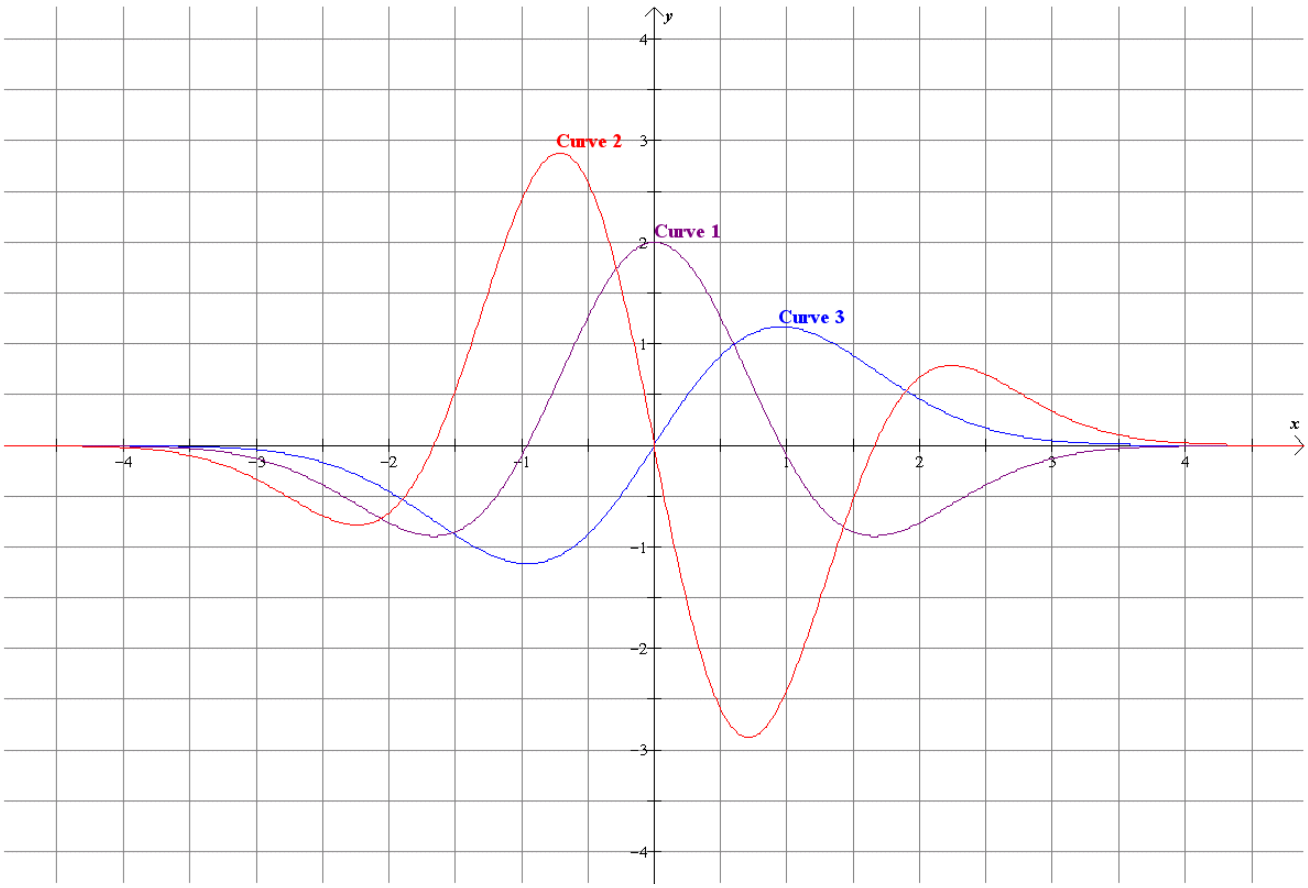


Self-Assessment: (2 bonus points)

a) Describe three strengths in your perfomance on this project. Include why each is a strength.

b) What are three things that could be improved about your performance on this project. Explain specifically how you will make these improvements.

c) Identify two things about this project which are still unclear to you.

d) Identify two insights that you have acquired in doing this project.

Name ______________________________ **Due 3/26/09**

Problems 1 through 6 are each worth 2 points. In each problem find and indicate for the given function

i) all real roots (zeroes)

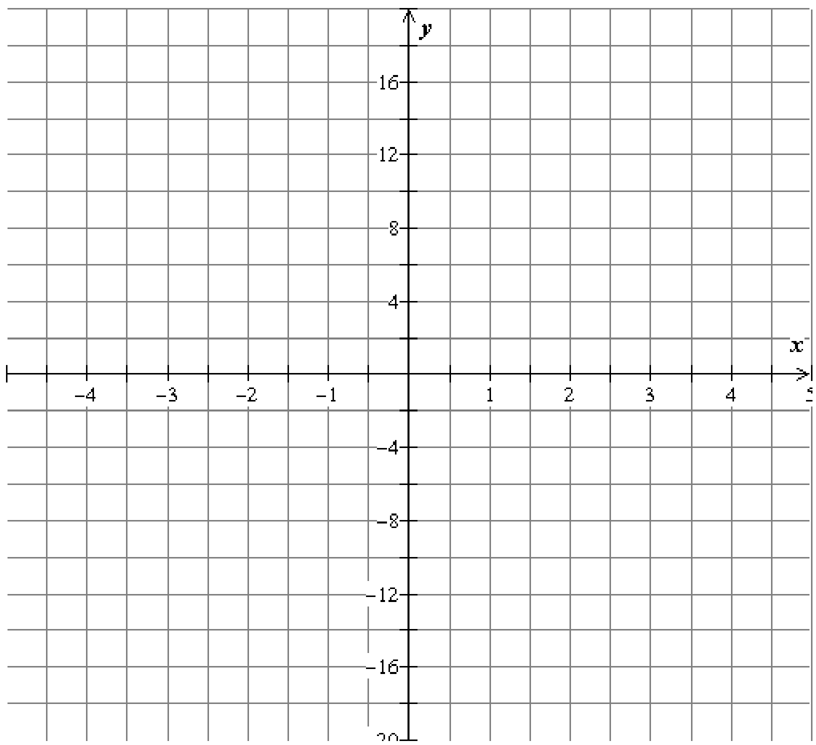
ii) any asymptotes

iii) the coordinates of any maxima or minima (and indicate whether they are absolute or local)

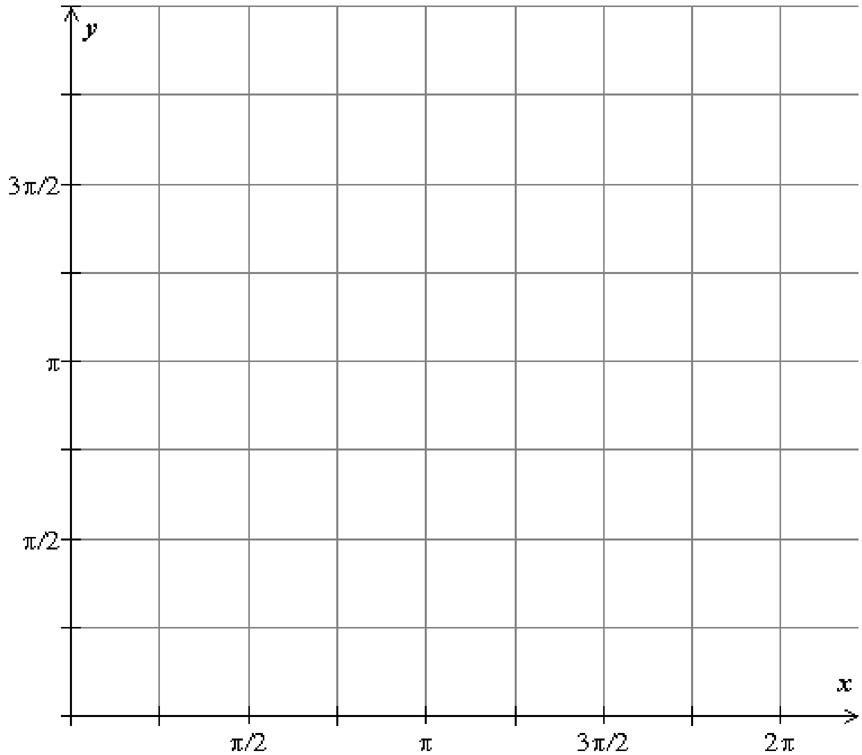
iv) any points of inflection

then sketch the function (or provide a computer generated graph). A graphing calculator or **WinPlot** may prove helpful in finding roots and graphing the functions.

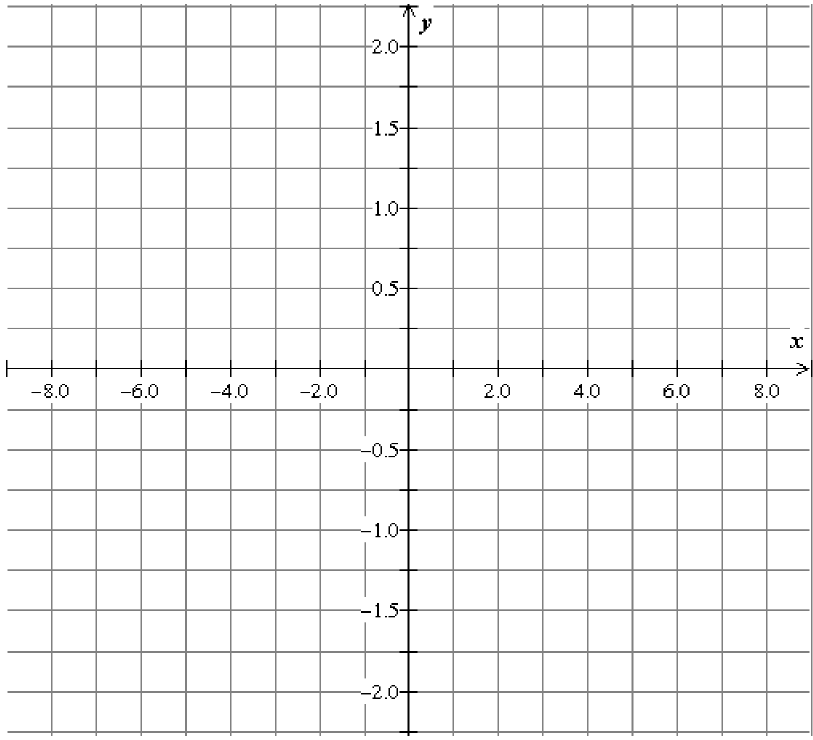
1. $f(x) = x^3 + 4x^2 - 2x - 3$



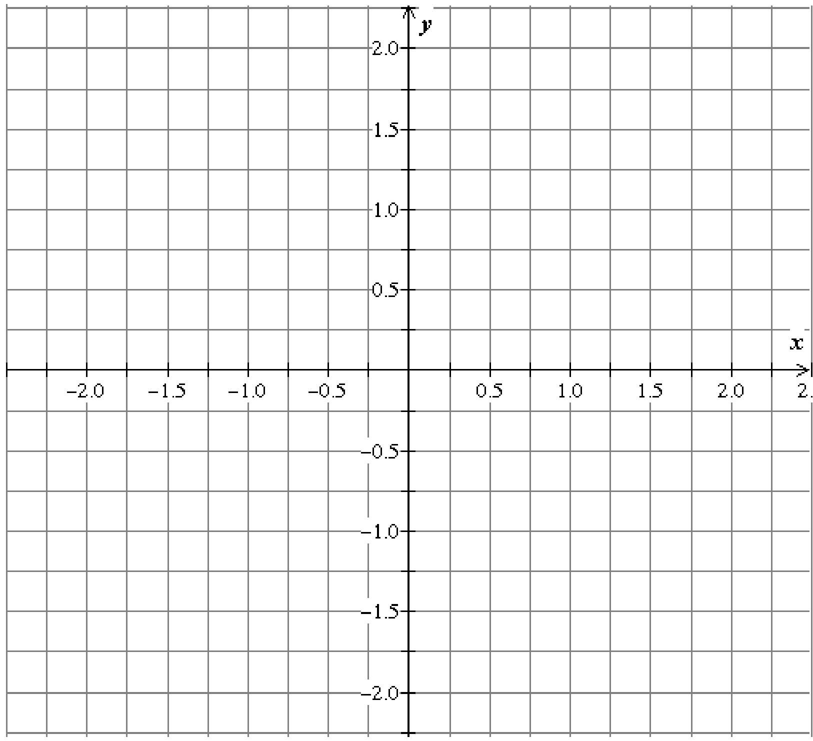
2. $f(x) = \dfrac{x}{\sqrt{2}} + \sin x$ on $[0, 2\pi]$



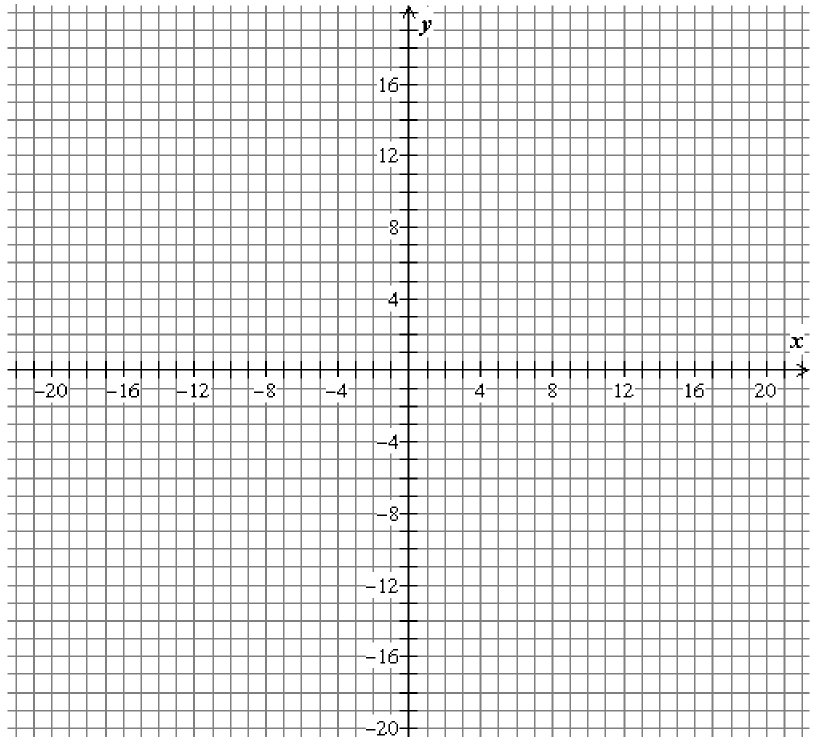
3. $f(x) = x^3 e^{-x}$



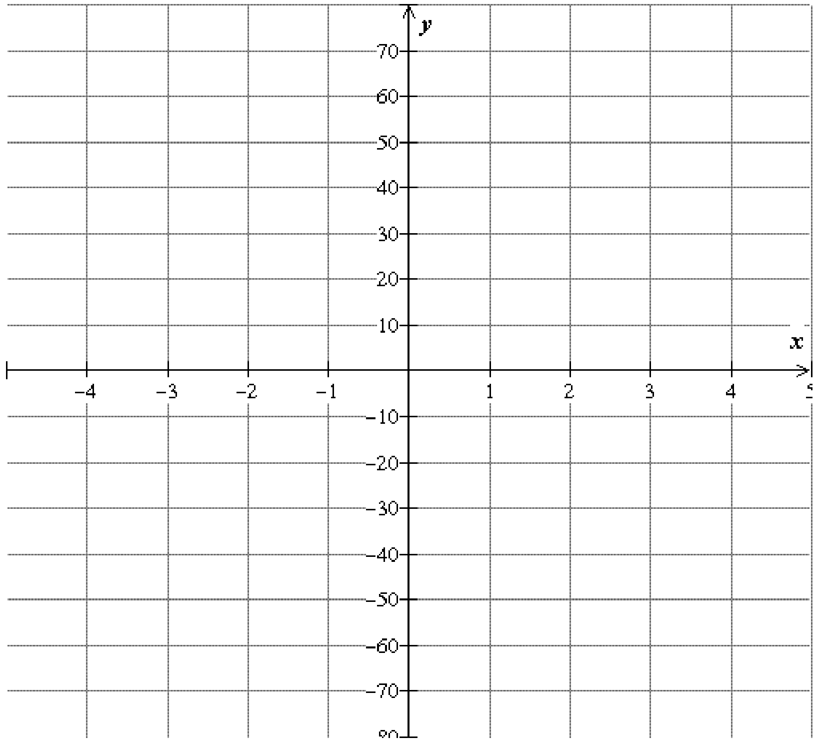
4. $f(x) = x^3 e^{-x^2}$



5. $f(x) = \frac{x^2-2x+1}{x+2}$



6. $f(x) = \frac{9x^4 - 82x^2 + 9}{x^2}$



7. (1 point)
A shallow barge is being drawn towards a dock by a taut (not sagging) cable attached to the top of a dock 6 ft above the deck of the barge. When the barge is 24 ft away from the bottom (water level) of the dock, it is moving in the water at a speed of 3 ft per second towards the dock. At this time, how fast is the cable being pulled in?

8. (2 points)
Two points are marked on the surface of a spherical balloon. One point is marked at the 'North Pole', while the other is on the 'Equator'.
a) In terms of the sphere's radius, r , what is the distance, s, between the two points as measured on the surface of the sphere? (i.e., What is the shortest length of string one can lay on the sphere between the two points?)

b) In terms of the sphere's radius, r , what is the distance, D, between the two points as measured in three dimensional space? (i.e., What is the shortest length of string one can position within the sphere between the two points?)

Air is pumped into the balloon in such a way that the balloon's volume is increasing at a constant rate of $3000\ \frac{\text{cm}^3}{\text{min}}$
c) When $r = 10.0\,\text{cm}$, how fast is s increasing?

d) When $r = 10.0\,\text{cm}$, how fast is D increasing?

9. (2 points)
Two street lights of height L are a distance D apart. The light at the top of one is functioning, but the other is being worked on by a tired repair person, who resents having to be there at night. If the repair person drops a wrench from rest at the top of the lamp pole, how fast is the wrench's shadow (made in the light of the other street lamp) moving horizontally along the ground when it is a height , h , above the ground? **Hint** : In the absence of air resistance, conservation of energy requires that $\frac{1}{2}v^2 + gh = gL$ where v is the downward velocity of the wrench, and g is a constant, known as the acceleration of gravity. Your answer should be a function of g, L , D, and h .

10. A rectangle is inscribed in the ellipse $\frac{x^2}{16} + \frac{y^2}{25} = 1$ with its sides parallel to the coordinate axes. (2 points)
a) Find the dimensions of the rectangle of largest area.

b) Find the dimensions of the rectangle of largest perimeter.

11. A rectangle is inscribed between the x axis and the stated curve. In each case find the dimensions of the rectangle with the greatest area. (2 points)

a) $y = e^{-x^2}$

b) $y = \begin{cases} 0 & \text{for } x < 0 \\ e^{-x} & \text{for } x \geq 0 \end{cases}$

12. A right circular cylinder is inscribed in a sphere of radius a. Find the largest possible volume of the cylinder. (2 points)

13. (1 point)
A simple model for the mutual potential energy, V, of a pair of inert gas atoms is the Lennard-Jones function,

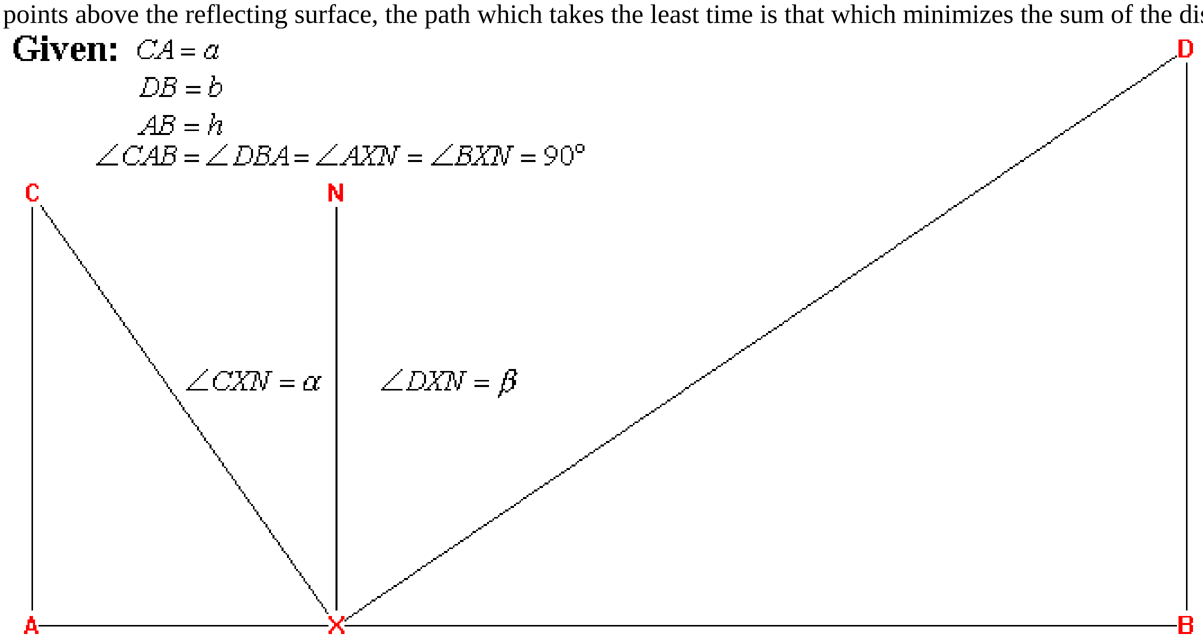
$$V = E\left(\left(\frac{a}{r}\right)^{12} - 2\left(\frac{a}{r}\right)^{6}\right)$$, where r is the distance between the atoms and E and a are positive constants (E having units of energy, while a has units of distance).

a) At what distance between the atoms is the potential energy a minimum?

b) What is the value of the potential energy at this position?

c) The force [directed towards increasing r (to right) if positive, and directed towards decreasing r (to left) if negative] between the two atoms is given by $-\frac{dV}{dr}$. The position of minimum potential energy is commonly referred to as the 'equilibrium position'. Explain this terminology.

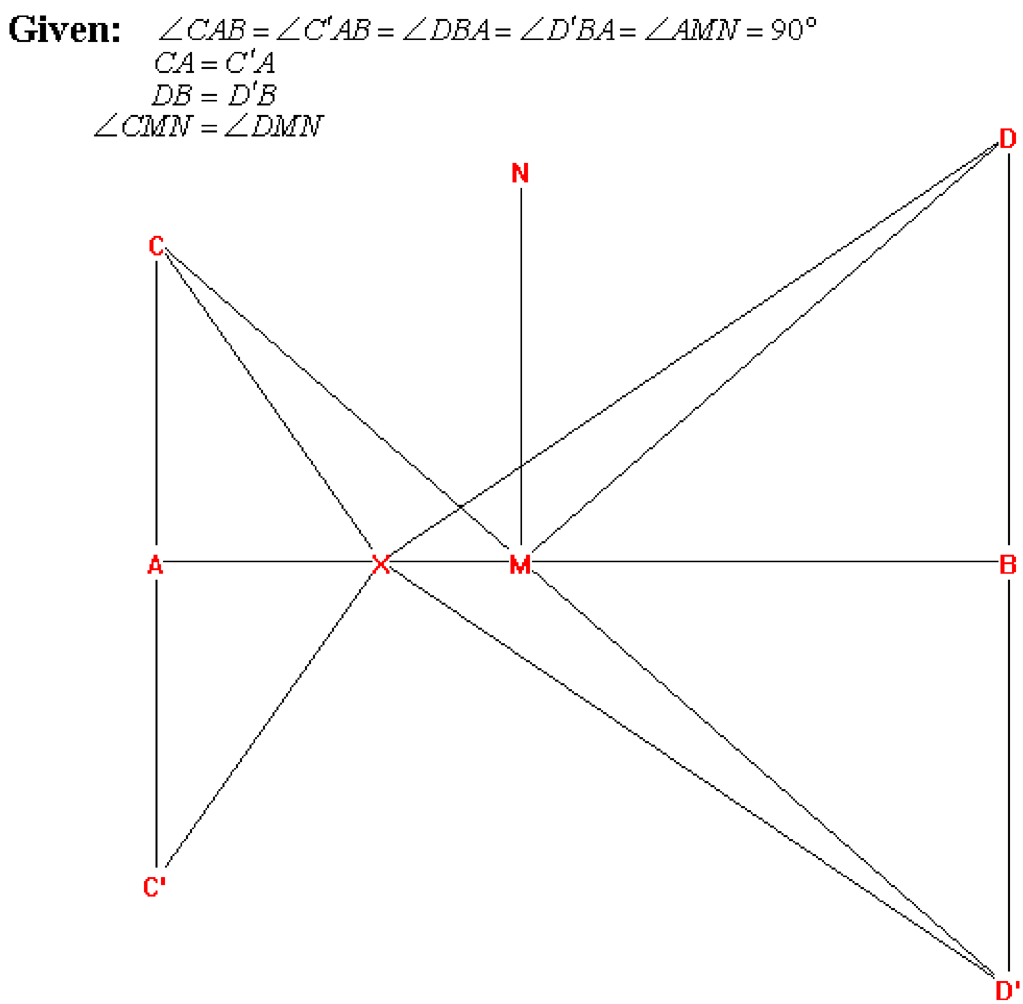
14. (3 points)
Fermat's principle of least time states that the path followed by a light ray in travelling between two points is the one that takes the least amount of time. Consider a light ray that gets from point C, a distance a above a plane refecting surface, to point D, a distance b above the same surface, by refecting off of that surface at point X. Let LN (the normal) be perpendicular to AB. The angle CXN $= \alpha$ is called the angle of incidence, while the angle DXN $= \beta$ is called the angle of reflection. Since the speed of light is constant for all points above the reflecting surface, the path which takes the least time is that which minimizes the sum of the distances CX + XD .



a) Determine an **analytic** expression for CX + XD in terms of the distances a and b and the angles α and β.

b) Determine the relation between α and β that minimizes CX + XD subject to the constraint that AB $= h$ is a constant.

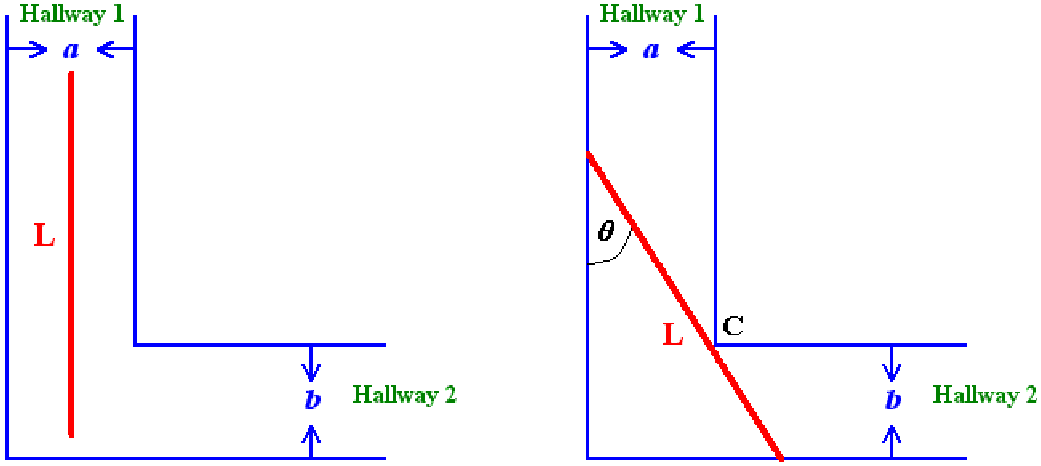
c) The relation between α and β that minimizes CX + XD can also be determined by a **synthetic** argument using elementary geometry. Present such an argument. [**Hint:** Consider the following diagram.]



Do one of the following three problems. You may do **all three** for six **bonus** points.

15. (3 points)

Two painters proceed down Hallway 1, which is of width a, carrying a ladder of fixed length L. At the end of this hallway is a right angle turn into Hallway 2 of width b. Assuming that the ladder is rigid (i.e., it does not bend), the minimum value for L, L_0, for which the ladder "just gets stuck" is also the maximum value of L for which the ladder just clears the corner!



a) Explain what is meant by the last statement which, superficially at least, appears to be a contradiction.

b) Determine an expression for L_0 expressed as a function of a and b. **Hint:** Imagine that you start with too large a ladder so that in attempting to get into Hallway 2 you get stuck against corner **C**. Determine the relationship between the angle $\boldsymbol{\theta}$ and L. Using implicit differentiation determine the critical value of L for which a small decrease in L would cause "an enourmous change" in $\boldsymbol{\theta}$.

c) For a ladder of fixed length L determine the minimum value of a expressed as a function of L and b that would allow the ladder to turn the corner.

d) For a ladder of fixed length L determine the minimum value of b expressed as a function of L and a that would allow the ladder to turn the corner.

e) The formula for L_0 that you developed in part b) is actually too big for a real ladder. Explain why.

f) Actually the formula for L_0 that you developed in part b) is too small for a real ladder. The painters could still manage to get around the corner with a ladder greater than L_0. Explain how they could do this.

16. (3 points)
A common problem of calculus instructors is coming up with ' nice ' examples of polynomial functions.
a) Consider a generic quadratic function of the form $f(x) = x^2 + cx + d$. Here c and d are integers. How many inflection points does $f(x)$ have ?

How many real critical points (turning points and/or zeros of $f'(x)$) are there for different choices of c and d ?

Are the critical points maxima, minima or neither?

How could you choose c and d to insure that all critical points occur at integer values of x ?

b) Consider a generic cubic function of the form $f(x) = x^3 + bx^2 + cx + d$. Here b , c and d are integers. How many inflection points does $f(x)$ have ?

Below give examples of a cubic function $f(x)$ where all the critical and inflection points occur at integer values of x and $f(x)$ has the stated number of real critical points.

No real critical points.

Exactly one real critical point.

Two real critical points.

17. (3 points)
The angle of deflection of a ray of light scattered by a spherical raindrop is given by the formula: $D(\alpha) = 2\alpha - 4\beta + \pi$, where α is the angle of incidence of the light to the raindrop and β is the angle of refraction of the light within the raindrop. By definition both angles are on the interval $\left[0, \frac{\pi}{2}\right]$. The two angles are related by Snell's law , $\sin\alpha = n\sin\beta$, where n is the index of refraction of water ≈ 1.33 .

a) Evaluate $D(0)$.

b) Evaluate $D'(0)$.

c) Explain whether D is an increasing or decreasing function of α.

d) Find an expression for the angle α_0 at which D is a minimum.

e) For $n = 1.33$ evaluate with a calculator $\pi - D(\alpha_0)$. This so called 'Rainbow Angle' is the angle from the horizontal at which an observer with her/his back to the sun should see a rainbow near dusk or dawn.

Self-Assessment: (2 bonus points)
a) Describe three strengths in your perfomance on this project. Include why each is a strength.

b) What are three things that could be improved about your performance on this project. Explain specifically how you will make these improvements.

c) Identify two things about this project which are still unclear to you.

d) Identify two insights that you have acquired in doing this project.

Instructions for the Group Project

The following guidelines should be adhered to in forming your group, doing the work and writing up the project.

Group Requirements:

Each group must consist of at least two individuals but no more than four individuals. You are free to form your own groups, but if you can't find a partner see me and I'll assign you to a group. Some class time will be devoted to group work, but much of it will have to be done outside of class. It is up to the group to decide any internal division of labor, eg., who is responsible for what parts of the problem, who will be the 'algebra expert', who will check the work, who will write up the what parts of the report. It is possible one group has one individual write the entire report, while in another group everyone writes up a different part of the report. It is in your own best interest to insist that you understand the solution of the whole problem in this project! You are free to use any written resources or computing technology in solving the problem.

Report Requirements:

Each group must hand in one report for a given project which should include the following :

1. The names of all group participants. If the report writers feel an individual did not perform his/her assigned task, you are free to delete that person's name from the report. I will arbitrate all appeals on such disagreements and reserve the right to give either a written or oral exam to decide the issue.

2. The conclusions of any questions stated neatly in complete sentences which are both concise and complete in expressing the answers.

3. The mathematical work to each problem attached in a way which is both neat and clear. Solutions should be presented in the report in the same order that the associated problems appear in the project.

Grading:

1. Each person in the group will receive the same point total out of 35 that the project report receives. Appeals on this are permitted, but I reserve the right to then administer either an oral or written exam to such an individual to replace the group score. Thus, it is the responsibility of everyone in the group to review the answers to all of the questions.

2. Grades will be based on the mathematical correctness both of the results and the methods used to arrive at them. Thus, a right answer arrived at by accident using faulty mathematics will not count for much. Points will be deducted for incomplete, illegible, sloppy or incomprehensible answers.

 Calculus I : Group Project /35

Name ______________________________ **In Class 3/30/09 & 3/31/09** **Due 4/08/09**

Name ______________________________

Name ______________________________

Name ______________________________

Choose one of the following two problems. You may earn 15 bonus points by doing both problems.

1. A sheet of rectangular paper has width, w , and length, l , with $w \leq l$. The sheet is positioned so that the short side is horizontal and the long side vertical. Let the four vertices of the rectangle be labeled A , B , C and D, with A the top left corner and the remaining vertices labeled clockwise starting from A. The upper left corner is folded over to touch the right vertical side at point Q on segment BC. A crease of length y is formed along the line segment PR, where P is the point on the top (segment AB) where the fold begins and R is the point where the fold ends. Let x stand for the length of segment AP. Starting from the smallest possible value of x the point R begins on the bottom (segment *CD*) and moves to the left. Eventually at a particular value of x called x_c , the point R coincides with the vertex D. For x greater than x_c point R is along the left vertical side (segment DA).

a) What are the allowed values (the domain) of x ?

b) Express x_c in terms of l and w . [**Hint :** Let $\theta = \angle APR$. Use trig identities to express $\cos(\pi - 2\theta)$ in terms of $\sin(\theta)$. To arrive at a formula for x_c you will have to solve a quadratic equation. Since $l > w$, only one root is in the proper domain of x .]

c) Express y in terms of x , l and w for $x \leq x_c$.

d) Express y in terms of x , l and w for $x > x_c$.

e) Considering y as a function of x, is y continuous at $x = x_c$? Explain your answer.

f) For $x < x_c$ y is an increasing function of x. For what values of the ratio $\frac{w}{l}$ does y begin to decrease for $x > x_c$?

g) In general, does $\frac{dy}{dx}$ exist at $x = x_c$? Explain your answer.

h) For what values of the ratio $\frac{w}{l}$ can the length of the crease be smaller than l ?

i) What is the length of the smallest crease possible for an $8.5''$ by $11''$ sheet of paper?

j) What is the length of the smallest crease possible for a $6''$ by $25''$ piece of paper?

2. A trucking company has hired you to determine the optimal highway speed, v , they should require of their drivers. The decision is to be based solely on economic grounds. The only two factors you need to consider are the driver's hourly wage, W , and the price of fuel (measured in $ per gallon), P . The company has provided you with data on highway speed versus the rate of fuel consumption, r . By doing a statistical least squares analysis you come up with the following linear model :

$r = \frac{8.34 \text{ miles}}{\text{gallon}} - \frac{0.0641 \text{ hours}}{\text{gallon}} v$, where r is measured in miles per gallon and v is measured in miles per hour.

The company would like the following table completed with the optimal speeds stated to the nearest mph. Because of legal requirements you can not recommend a speed in excess of 65 mph.

Table of Optimal Speeds to the Nearest mph

	P (Price of Fuel in dollars per gallon)				
W ($/hr)	2.00	2.50	3.00	3.50	4.00
11.0					
12.0					
13.0					
14.0					
15.0					
16.0					
17.0					

In addition to the completed table, the company wants a thorough explanation of how you arrived at your results with enough detail provided so that they could calculate the optimal speed for any new values of W and P .

Name ______________________________ **Due 4/20/09**

1. (7.5 points)

Evaluate the following integrals. Evaluate the following integrals. The Wolfram Research On Line Integrator at http://integrals.wolfram.com/index.jsp evaluates indefinite integrals symbolically, but it does not supply the arbitrary constant, hence its results may differ by an algebraic rearrangement from any other correct answer. The **WinPlot** program or a graphing calculator can evaluate definite integrals numerically. To use **WinPlot** to evaluate the definite integral, $\int_a^b f(x)dx$ use the 2-dim Window. Enter the formula for the integrand, i.e., $f(x)$, under the 1. Explicit y = f(x) Equa format. From the View menu choose Vew/Set corners to set left less than a and right larger than b. The down and up values depend on the extremes of $f(x)$ on $[a, b]$. From the View menu, Grid may be chosen to set convenient tick marks and tick labels. To numerically evaluate the definite integral choose Measurement/Integration from the One menu. Set the lower limit to a and the upper limit to b. Choose the desired number of subintervals (i.e., the n value for the numerical quadrature), select the approximation method to be used (left endpoint, midpoint, right endpoint, trapezoidal, parabolic (Simpson's rule) and random) and then press the definite button to see the numerical result. Answers to indefinite integrals can also be checked this way by seeing if the calculated value of the associated definite integral with chosen upper and lower limits agrees with the numerical definite integral (see **http://www.execpc.com/~aplehnen/calc2pr2.htm**).

a) $\int (15x + 3)dx$ = ______________________________

b) $\int (15x + 3)^{-3}dx$ = ______________________________

c) $\int \frac{8t+1}{\sqrt{3t + 12t^2}}dt$ = ______________________________

d) $\int \sin(x)\cos^4(x)dx$ = ______________________________

e) $\int \frac{\sec^7(\sqrt{x})\sin(\sqrt{x})\,dx}{\sqrt{x}}$ = ______________________________

f) $\int \frac{\sin^7(x)}{\cos^7(x)}\,dx$ = ______________________________

g) $\int_0^3 \left(\sqrt{x+1} - \frac{1}{\sqrt{x+1}} \right) dx$ = ______________________________

h) $\int_0^3 \frac{x\,dx}{\sqrt{x+1}}$ $=$ ______________________

[**Hint :** look closely at g) !]

i) $\int_1^{\mathrm{e}} \frac{\ln(x^8)dx}{x}$ $=$ ______________________

j) $\int_0^{\frac{\pi}{2}} e^{-\sin(x)}\cos(x)dx$ $=$ ______________________

k) $\int \frac{e^{4x}}{1+e^{4x}}dx$ $=$ ______________________

l) $\int \frac{x^3\sin\left(x^4\right)}{\cos^2\left(x^4\right)}\,dx$ $=$ ______________________

m) $\int e^{-x}\sin(1+e^{-x})\sec^3(1+e^{-x})dx$ $=$ ______________________

n) $\int_0^1 \frac{\cosh(x)}{1+\sinh(x)}dx$ $=$ ______________________

o) $\int \frac{\cot\left(\sqrt{x}\,\right)}{\sqrt{x}}dx$ $=$ ______________________

2. Given the following functions, compute the derivatives: (2 points)

a) $F(x) = \int_0^x \frac{dt}{\sqrt{6t+1}}$ $F'(x) =$ ______________________

b) $G(x) = \int_0^{\cos x} \frac{t^6 dt}{1-t^2}$ $G'(x) =$ ______________________

c) $H(x) = \int_{x^2}^{4x^2} \frac{dt}{\sqrt{t} + t}$, for $x > 0$, $H'(x) =$ ______________________

d) $S(\omega) = \int_0^{\sqrt{\omega}} \frac{\sin(t)}{t} dt$ $S'(\omega) =$ ______________________

3. Let $f(x) = \int_0^x e^{-4t^2} dt$ (1 point)

a) State the linear (tangent) approximation to $f(x)$ about $x = 0$.

b) Evaluate the limit : $\lim_{x \to 0} \frac{f(x)-x}{x^3}$ $=$ ______________________

4. Evaluate the following limits by equating them to definite integrals: (3 points)

a) $\lim_{n \to \infty} \sum_{i=1}^{n} \frac{\pi}{n} \cos\left(\frac{i\pi}{n}\right)$ $=$ ______________________

b) $\lim_{n \to \infty} \sum_{j=1}^{n} \frac{2}{n} \left(4 + \frac{2j-2}{n}\right)^4$ $=$ ______________________

c) $\lim_{n \to \infty} \sum_{k=1}^{n} \frac{8k}{n^2} e^{-\left(\frac{2k}{n}\right)^2}$ = ______________________

5. (1 point) A student uses the Fundamental Theorem of Calculus to calculate $\int_{-1}^{1} \frac{dy}{y^2} = -\frac{1}{y}\Big]_{-1}^{1} = -\frac{1}{1} - \left(-\frac{1}{-1}\right) = -2$

a) Is this result reasonable? Explain.

b) Does the Fundamental Theorem of Calculus apply to this integral? Explain.

6. Solve the following differential equations subject to the given conditions: (3 points)

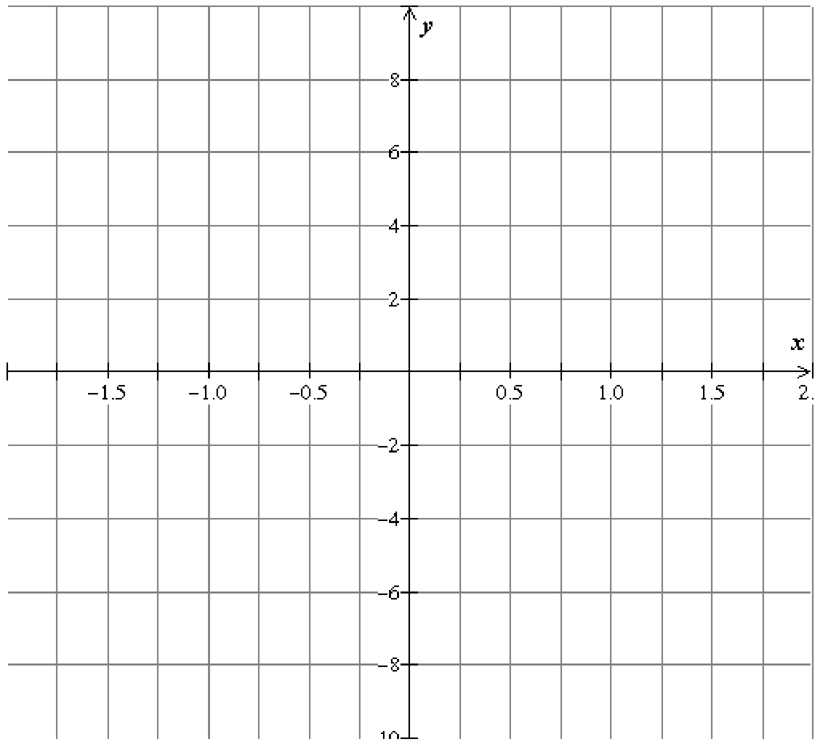
a) $y'(x) = 6x + 1$ with $y = -3$ when $x = 1$

b) $\frac{dy}{dx} = \frac{1}{\sqrt{2x+1}}$ with $y = -\frac{1}{4}$ when $x = 4$

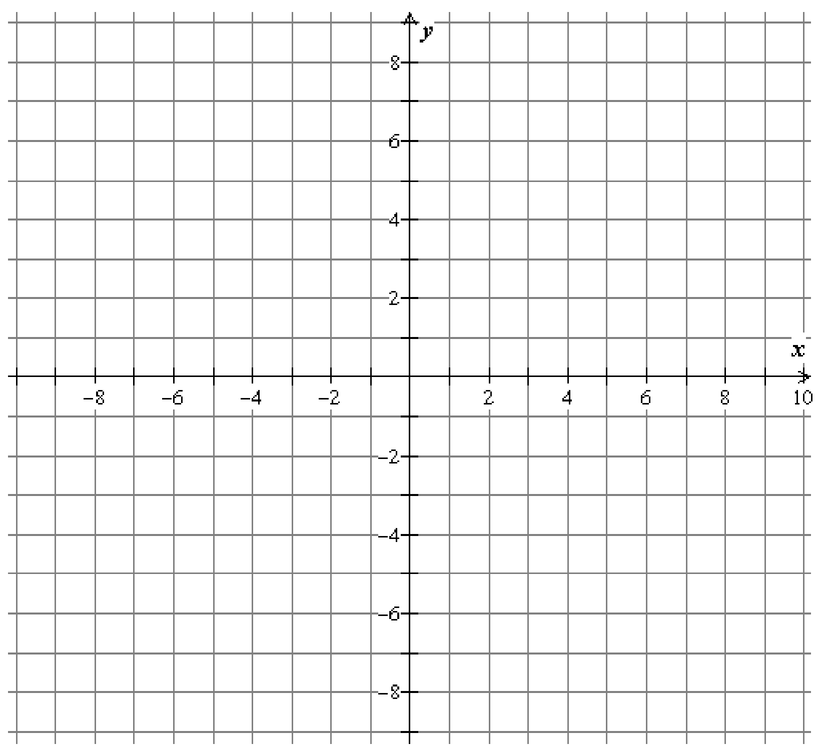
c) $f''(x) = -\cos(x)$ with $f(0) = 3$ and $f'(0) = -2$

7. Solve the following first order differential equations and sketch the solutions: (4 points)

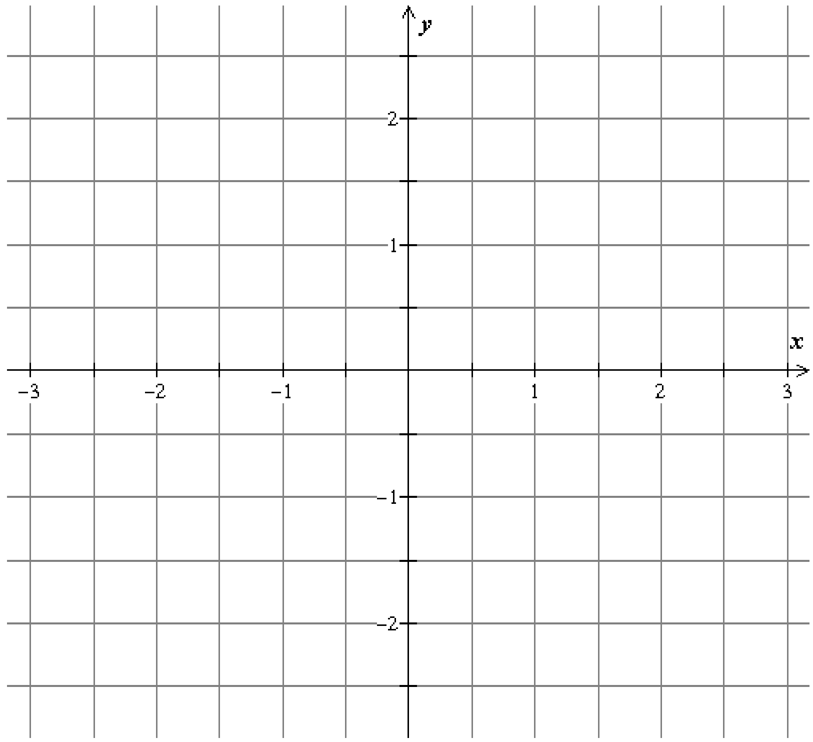
a) $\frac{df}{dx} = -\frac{1}{\sqrt{4-x^2}}$ with $f(0) = 6$



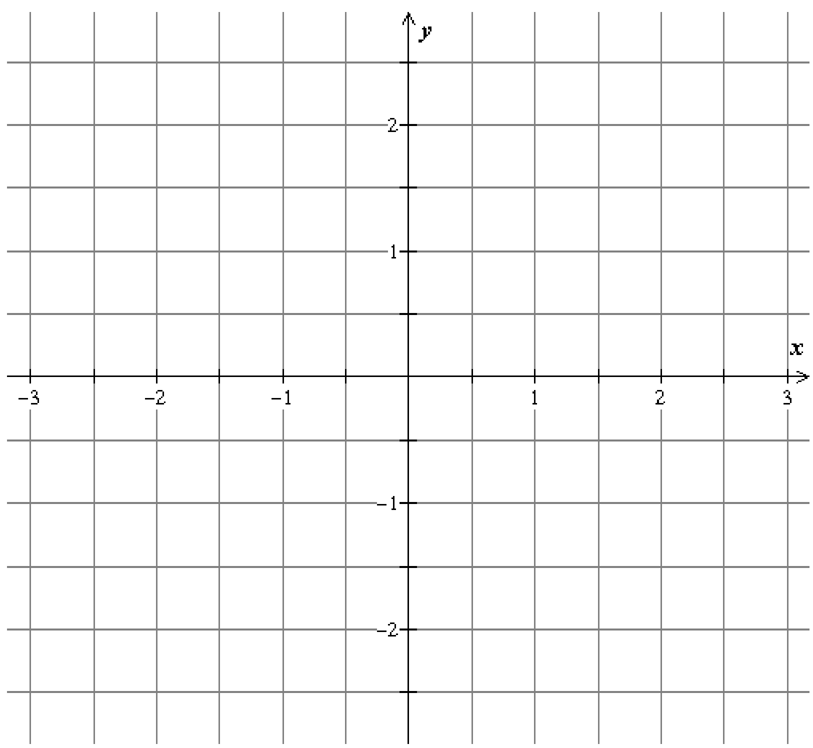
b) $\frac{df}{dx} = \frac{12}{9x^2+4}$ with $f(0) = 2$



c) $\frac{dy}{dx} = -y(x-1)$ with $y(1) = -2$



d) $f'(x) = -2f(x)$ with $f(0) = 2$



Linear Circuits (Each of the following two problems is worth 2 points)

8. A constant resistor R and a constant inductor L are hooked up in series to a DC power supply which outputs a constant potential of V . The electric current i builds up in a manner described by: $Ri + L\frac{di}{dt} = V$ with $i = 0$ at $t = 0$.

a) Verify that $i = \frac{V}{R}(1 - e^{-\frac{Rt}{L}})$ gives the current as a function of the time t .

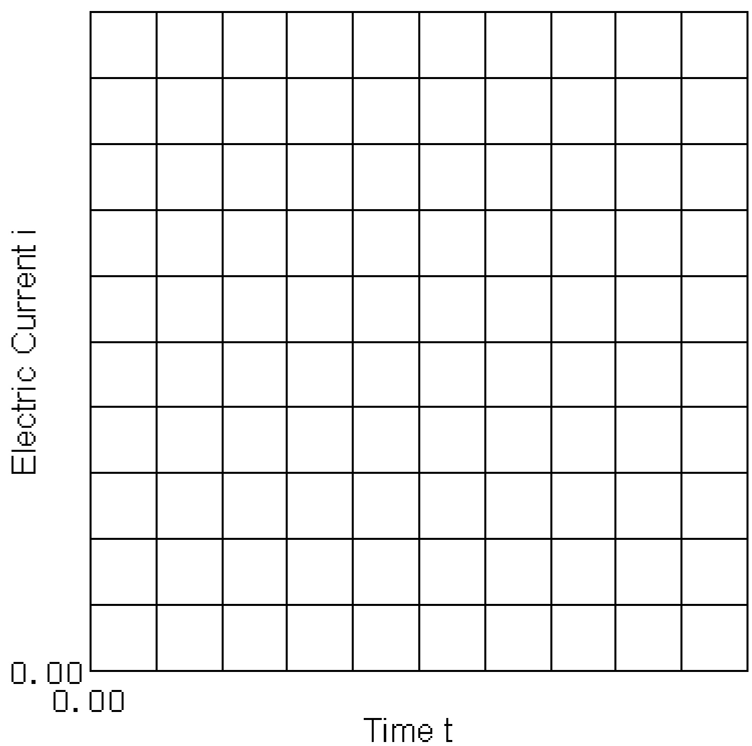
b) What combination of R and V sets the relevant electrical current scale for this problem, i.e., how should the current axis of an i versus t graph be marked?

c) What combination of R and L sets the relevant time scale for this problem, i.e., how should the time axis of an i versus t graph be marked?

d) If $R = 6.00\,\Omega$, $L = 0.300\,$H, and $V = 24.0\,$V , how long after the power supply is switched on does it take the current to reach 1.00 A ? **Note :** $\frac{1\text{H}}{1\,\Omega} = 1$ second and $\frac{1\text{V}}{1\,\Omega} = 1$ A.

e) For arbitrary values of R , L and V , how long after the power supply is switched on does it take the current to reach $\frac{V}{4R}$?

f) For arbitrary values of R , L and V sketch a graph of i versus t. Label each axis in units of the relevant scale factors.



9. A constant capacitor C is charged by a DC power supply to a fixed potential difference V_0 . The power supply is then removed and the capacitor is allowed to discharge through a constant resistor R . The charge voltage drop across the capacitor, V, decays in a manner described by : $R\frac{dV}{dt} + \frac{V}{C} = 0$ with $V = V_0$ at $t = 0$.

a) Solve this differential equation for voltage across the capacitor as a function of the time t .

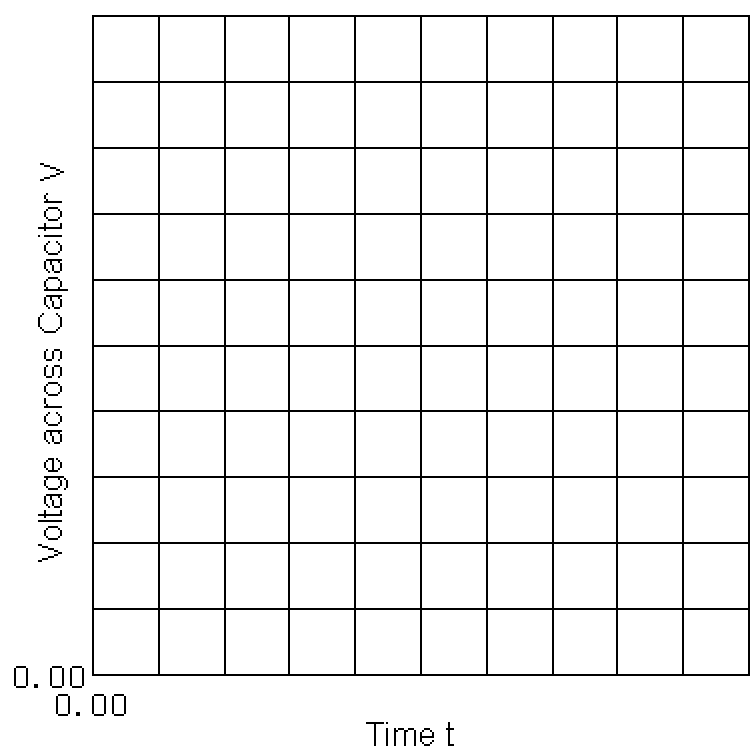
b) What sets the relevant scale for the voltage drop across the capacitor in this problem, i.e., how should the voltage drop across the capacitor axis of a V versus t graph be marked?

c) What combination of R and C sets the relevant time scale for this problem, i.e., how should the time axis of a V versus t graph be marked?

d) If $R = 2.00\,\text{M}\Omega$, $C = 1.00\,\mu$F, and $V_0 = 20.0\,\text{V}$, how long after the power supply is switched off does it take the voltage drop across the capacitor to reach $1.00\,\text{V}$? **Note** : $1\text{F} \times 1\,\Omega = 1\,\text{second}$.

e) For arbitrary values of R , C and V_0 , how long after the power supply is switched off does it take the voltage drop across the capacitor to reach $0.05V_0$?

f) For arbitrary values of R , C and V_0 sketch a graph of V versus t. Label each axis in units of the relevant scale factors.



10. (1.5 points)
Scotty, responding to captain's orders, engages engines in deep space and his starship propels forward with an acceleration given by $a = \frac{9500\,t \text{ meters}}{\text{second}^3}$, where t is the time in seconds. If at time $t = 0$, the starship was already traveling (with respect to the fixed stars) at a speed of $45,000 \ \frac{\text{m}}{\text{sec}}$ in the same direction as it is being accelerated.
a) How fast is the starship moving at the end of the first minute?

b) How far has it traveled at the end of the first minute?

11. (3 points)
a) Determine the **analytic** answer for the following definite integral: $\int_0^1 \frac{1}{1+x^2}\,dx =$ ______________________

b) Using a calculator determine the eqivalent eight decimal digit approximation of this analytic result. ______________________

In order to make the numerical quadrature below more painless, you should use a computer or calculator to perform the calculations.
To use **WinPlot** to evaluate the definite integral $\int_a^b f(x)dx$ use the 2-dim Window. Enter the integrand (i.e., $f(x)$) under the 1. Explicit y = f(x) Equa format. From the View menu choose Vew/Set corners to set **left** is less than a and **right** is larger than b . The down and up depend on the extremes of $f(x)$ on $[a,b]$. Also from the View menu Grid may be chosen to set convenient tick marks and tick labels. To numerically evaluate the definite integral choose Measurement/Integration from the One menu. Set the **lower limit** to a and the **upper limit** to b . Choose the desired number of subintervals (i.e., the n value for the numerical quadrature), select the approximation method to be used : left endpoint, midpoint, right endpoint, trapezoidal, parabolic (Simpson's rule) and random. For the Simpson's rule method **WinPlot** uses for the number of subintervals twice the number entered as n, so be sure to enter half of the requested value. Press the **definite** button to see the numerical result.

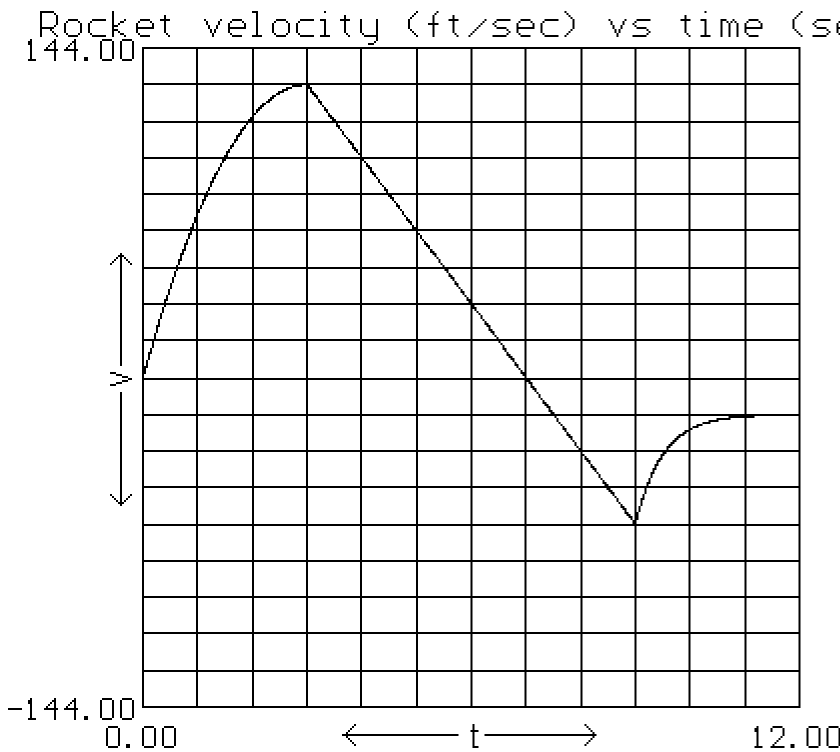
b) Now approximate this same integral using the Midpoint , the Trapezoidal and Simpson's Rule. For all three approximations start with 4 subdivisions of the interval of integration (i.e., $n = 4$) and keep doubling the number of subdivisions until either the approximate integral is within one one millionth of its last estimate or 128 subdivisions have been used.

n	Mid Point Rule	Trapezoidal Rule	Simpson's Rule
4			
8			
16			
32			
64			
128			

c) Which of the three approximations converges fastest to the correct answer? Explain why this is not surprising.

Bonus problems. Each problem is worth 3 points.

12. Below is the graph of the velocity of a model rocket versus the time since it took off.



a) How fast was the rocket moving when it took off?

b) At what time did the engine burn out?

c) At what time did the rocket reach its highest altitude?

d) At what time did the rocket's parachute deploy?

e) Estimate the rocket's terminal velocity after the parachute deployed.

f) Give a rough estimate of how high the rocket traveled. Explain your method.

g) Give a rough estimate of the time of flight of the rocket. Explain your method.

13. Population Growth

a) The most naive (and most frightening) model for population growth is the exponential growth model. This assumes that the rate of growth is directly proportional to the current population size. In symbols this is expressed as $\frac{dN}{dt} = kN$, where N is the number of individuals in the population at time t and k is a constant. Roughly, k is the number of new 'offspring' produced by each individual in a relevant unit of time. Solve $\frac{dN}{dt} = kN$, subject to the initial condition that $N(0) = N_0$.

b) The exponential model is obviously unrealistic for large times in that it leads to unbounded population growth. There is only so much matter in the universe! At some point limited food and living space limit the population size of any species. One way to model this behavior mathematically is to modify the rate equation as follows : $\frac{dN}{dt} = kN(1 - \frac{N}{L})$, where L is the limiting population size. Initially N increases exponentially as before; however, as N approaches L the rate of growth slows to zero. Hence, $N = L$ is a horizontal asymptote of the solution. This rate equation is sometimes called the logistic equation.

For the logistic equation model what value of N makes the rate of growth a maximum?

What is this maximum rate of growth? How does this compare to the rate of growth of the exponential model for the same population size?

Verify the algebraic identity that $\frac{1}{N(L-N)} = \frac{1}{LN} + \frac{1}{L(L-N)}$.

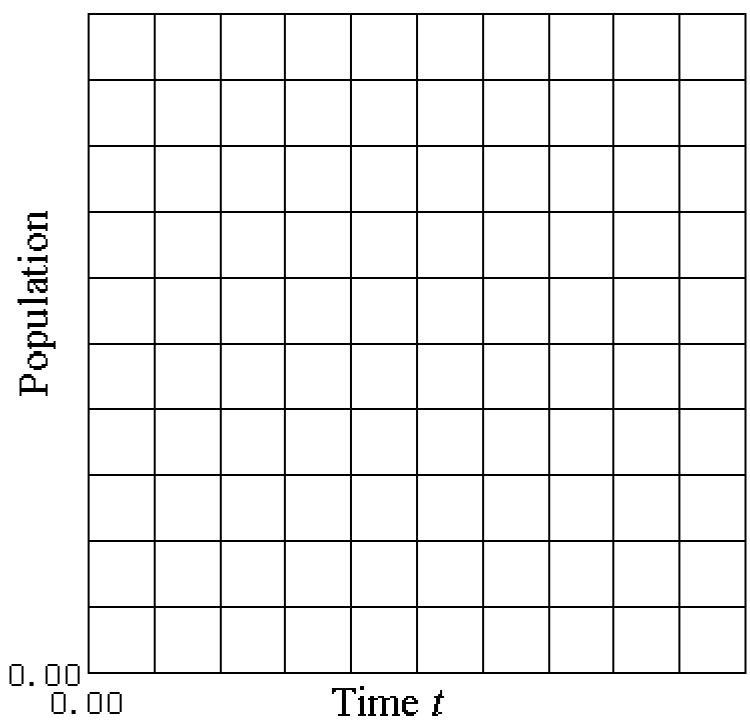
Using this identity solve the logistic equation, subject to the initial condition that $N(0) = N_0$.

Evaluate $\lim_{t \to \infty} N(t) =$ ________________________________

For $N_0 = 400$, $L = 400,000$ and $k = 0.05$ year^{-1} , according to the exponential model how long would it take the population to double to 800 ? To increase to $300,000$?

For $N_0 = 400$, $L = 400,000$ and $k = 0.05$ year^{-1} , according to the logistic model how long would it take the population to double to 800 ? To increase to $300,000$?

For $N_0 = 400$, $L = 400,000$ and $k = 0.05$ year^{-1} , make a careful graph of the solutions of both the exponential and logistic models. A graphing calculator or computer program would be helpful here.



Self-Assessment: (2 bonus points)
a) Describe three strengths in your perfomance on this project. Include why each is a strength.

b) What are three things that could be improved about your performance on this project. Explain specifically how you will make these improvements.

c) Identify two things about this project which are still unclear to you.

d) Identify two insights that you have acquired in doing this project.

Name ____________________________ **Due 5/07/09**

WinPlot can both display and calculate volumes of solids of revolution. Choose the 2-dim Window and in the **Equa** menu pick **1. Explicit** y=f(x) and enter the formula for $f(x)$. From the **View** menu choose **Set** corners to set the relevant window for x and y. From the **One** menu select Revolve Surface. In the surface of revolution dalogue box choose the axis of rotation, $ax + by = c$, by designating values for $a, b,$ and c or click "x-axis" or "y-axis" for an automatic setup. For example, $a = 0,\ b = 1,$ and $c = 1$ rotates the graph about the line $y = 1$. Set arc start to the starting x value and arc stop to the final x value. Angle start and angle stop determine the amount of rotation. Use the 'default' values of angle start = 0 and angle stop = 2pi. Press on **see surface** to display a 3 D graph of the solid. The arrow keys allow you to rotate your point of view. Every new request is added to the same 3D window, so you can see multiple examples simultaneously.

1. (2 points)

a) Calculate the area bounded by the curves $y = \dfrac{x^2}{27}$ and $x = y^2$.

b) What is the volume of the solid generated when the region in part a) is rotated about the y axis?

2. (2 points)

a) Calculate the volume of the solid generated when the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is rotated about the x axis.

b) What is this solid when $a = b$?

c) What is your answer to part a) when $a = b$? Is this the correct formula?

3. (3 points)

a) Find the volume of the solid generated when the curve $y = \dfrac{1}{x}$ from $x = 1$ to $x = e^a$ is rotated about the x axis.

b) What is the area under the curve $y = \frac{1}{x}$ from $x = 1$ to $x = e^a$? (This is half of the area of the xy plane cross section of the solid.)

c) Set up but don't evaluate (**Note**: If you are curious, the trig substitution $x^2 = \tan(\theta)$ works.) the definite integral which gives the lateral surface area of this solid.

d) What is the limit of the solid's volume as $a \to \infty$?

e) What is the limit of the xy plane cross sectional area as $a \to \infty$?

f) Do these results seem peculiar? Explain.

4. (1 point) A basin is in the shape of a hemisphere of radius r. Let h represent the depth of water at the bottom of the basin. Determine the volume, V, of water in the basin as a function of depth, h .

5. Consider the curve $y = 2\sin(x)$ from $0 \leq x \leq \pi$. (2 points)
a) What is the volume of the solid generated when this curve is rotated about the line $y = 0$?

b) What is the volume of the solid generated when this curve is rotated about the line $y = 2$?

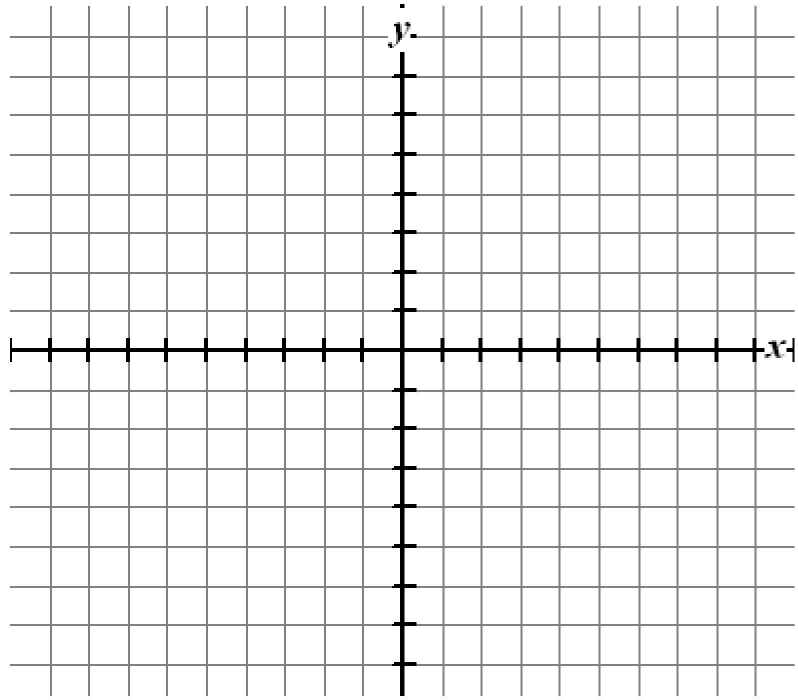
6. Consider the region in the first quadrant bounded by the curves $y = e^{-2x}$, $x = 0, y = 0,$ and $x = 1$. (3 points)
a) What is the volume of the solid generated when this region is rotated about the line $y = 0$?

b) What is the volume of the solid generated when this region is rotated about the line $y = 1$?

c) What is the volume of the solid generated when this curve is rotated about the line $x = 0$?
[**Hint** : What is $\frac{d}{dx}\left(-\frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x}\right)$?]

7. (4 points)

a) Sketch the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, $a > 0$.



b) Calculate the total length of the astroid.

The astroid is rotated about the y axis to make a particularly "nasty looking" solid of revolution.

c) Calculate the surface area of this solid of revolution.

d) Calculate the volume of this solid of revolution.

8. (1 point)
The current, i , of an AC power supply as a function of time, t , is given by $i(t) = i_0 \sin(\omega t + \delta)$, where i_0 , ω , and δ are constants. The instantanious power delivered to a load of resistance R is given by $P = i^2 R$. Over one period or cylce, what is the avearge power delivered?

9. (1 point)
The force exerted by an 'ideal' spring is given by $F = -k(x - x_0)$, where k is a positive constant, $x - x_0$ is the displacement of the spring from its unstretched equilibrium position at $x = x_0$.
a) How much work is done in compressing the spring from a position at $x = x_0$ to a position at $x = a\,(a < x_0)$?

b) How much work is done in stretching the spring from a position at $x = x_0$ to a position at $x = a\,(a > x_0)$?

10. (1 point) A glass aquarium is in the shape of a right rectangular prisim (i.e., a box!). The bottom floor is 1 foot by 28 inches, the depth is 14 inches. When filled to capacity it holds 20.36 gallons. Assume there is water in the aquarium to a depth of 1 foot.
a) What is the force on the $12''$ by $14''$ face due to the water?

Force $=$ ______________________ lbs

Force $=$ ______________________ N

b) What is the force on the $12''$ by $28''$ floor due to the water?

Force $=$ ______________________ lbs

Force $=$ ______________________ N

11. (1 point) A circular glass window of radius a allows one to see inside a large metal holding tank. If the liquid filling the tank has a mass density of ρ and if the center of the window is at a depth D below the surface of the liquid with $D > a$, calculate the force on the window due to the liquid.

12. (4 points)
a) A liquid of mass density ρ fills a right circular cylinder of radius r and height h to a depth D $(D < h)$. Calculate the work done in pumping all of this liquid out the top of the cylinder.

b) For part a) what is the ratio of the work to pump out all of the liquid to the total mass of the liquid?

c) A liquid of mass density ρ fills an inverted (pointed end down) right circular cone of base radius r and height h to a depth D $(D < h)$. Calculate the work done in pumping all of this liquid out the top of the cone.

d) For part c) what is the ratio of the work to pump out all of the liquid to the total mass of the liquid?

13. (3 points) A trough of length L has symmetric parabolic cross sections perpendicular to its length. Each such perpendicular cross section has a vertex a distance D below the top of the trough and a width across the top of the trough of w.
a) If the trough is full of a liquid of mass density ρ and the acceleration of gravity is g, what is the force due to the liquid acting on one end of the trough?

b) Calculate the work done in pumping out all of the liquid from the trough to a height h above the top of the trough.

14. (3 points) For a a positive constant, the following cumulative distribution function $F(x)$ gives the probability that the value of the random variable is less than or equal to x,

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 - e^{-x/a} & \text{if } x > 0 \end{cases}$$

a) verify that $F(x)$ "makes sense" as a cumulative probability distribution.

b) determine the probability density function $f(x)$ = ____________________

c) determine the mean or expected value of x: $\mu_x = < x >$ = ____________________
Hint: calculate $\frac{d}{dx}\left[e^{-x/a}\left(-ax - a^2\right)\right]$

d) determine the expected value of x^2: $< x^2 >$ = ____________________
Hint: calculate $\frac{d}{dx}\left[e^{-x/a}\left(-ax^2 - 2a^2x - 2a^3\right)\right]$

e) determine the standard deviation σ_x = ____________________

Bonus problems. Each problem is worth 4 points.
15. When a plane intersects a sphere, it slices the sphere into two 'caps' . Take the radius of the sphere to be R. Consider the smaller of the two caps. (If neither is smaller the two caps are hemispheres.) Let r be the radius of the base of the cap, and let h be the height of the cap.
a) Solve for R in terms of r and h .

b) Which is larger r or h ?

c) Solve for the volume of the cap in terms of r and h .

d) Solve for the surface area (excluding the base area) of the cap in terms of r and h .

Now consider the larger of the two caps. Again let r be the radius of the base of the cap, and let h be the height of the cap.
e) Solve for R in terms of r and h .

f) Which is now larger r or h ?

g) Solve for the volume of the cap in terms of r and h .

h) Solve for the surface area (excluding the base area) of the cap in terms of r and h .

i) For a cap of fixed volume, V , solve for r^2 as a function of h and V.

j) For fixed volume what is the maximum value, h_{m} of h ?

k) ACME tent company wants to design a dome tent in the shape of a spherical cap. The tent must have a volume of $8\ \mathrm{m}^3$. The cost per unit area of the material for the tent floor is only $\frac{1}{4}$ that of the material for the rest of the tent. Determine the values of r and h that minimize the cost of the tent.

16. Suppose we were interested in calculating the average angle that a curve, $y = f(x)$, makes with the 'horizontal' (i.e., the direction of the positive x axis) in going from the point $\left(a, f(a)\right)$ to the point $\left(b, f(b)\right)$. One way to do this, of course, is to calculate the average based on sampling $n+1$ equally spaced x coordinates from $x = a$ to $x = b$. That is, one could calculate the following function:

$$\overline{\theta}\ (n) = \frac{\sum_{i=0}^{n} \theta\,(x_i)}{n+1}$$; where $x_i = a + \frac{(b-a)i}{n}$ and $\theta\,(x_i)$ is the angle the curve $y = f(x)$ makes with the horizontal line $y = f(x_i)$ at the point $\left(x_i\ ,\ f(x_i)\right)$.

Consider the following three curves:

Curve 1	$f(x) = \sin(x)$	from	$x = -\frac{\pi}{2}$	to	$x = \frac{\pi}{2}$
Curve 2	$g(x) = x^2$	from	$x = 0$	to	$x = 1$
Curve 3	$h(x) = 1 - \sqrt{1 - x^2}$	from	$x = 0$	to	$x = 1$.

Using either a calculator or a computer program calculate the average angles for the value of n specified and fill in the table below. Give the results to at least four decimal places.

[**Hint :** $$\overline{\theta}\ (n) = \frac{\left(\frac{n}{n+1}\right)\sum_{i=0}^{n} \theta\,(x_i)\Delta x}{(b-a)}$$, where $\Delta x = \frac{(b-a)}{n}$.]

Average Angle	**Curve 1**	**Curve 2**	**Curve 3**
$\overline{\theta}\ (2)$			
$\overline{\theta}\ (10)$			
$\overline{\theta}\ (20)$			
$\overline{\theta}\ (100)$			
$\lim_{n \to \infty} \overline{\theta}\ (n)$			

Self-Assessment: (2 bonus points)

a) Describe three strengths in your perfomance on this project. Include why each is a strength.

b) What are three things that could be improved about your performance on this project. Explain specifically how you will make these improvements.

c) Identify two things about this project which are still unclear to you.

d) Identify two insights that you have acquired in doing this project.