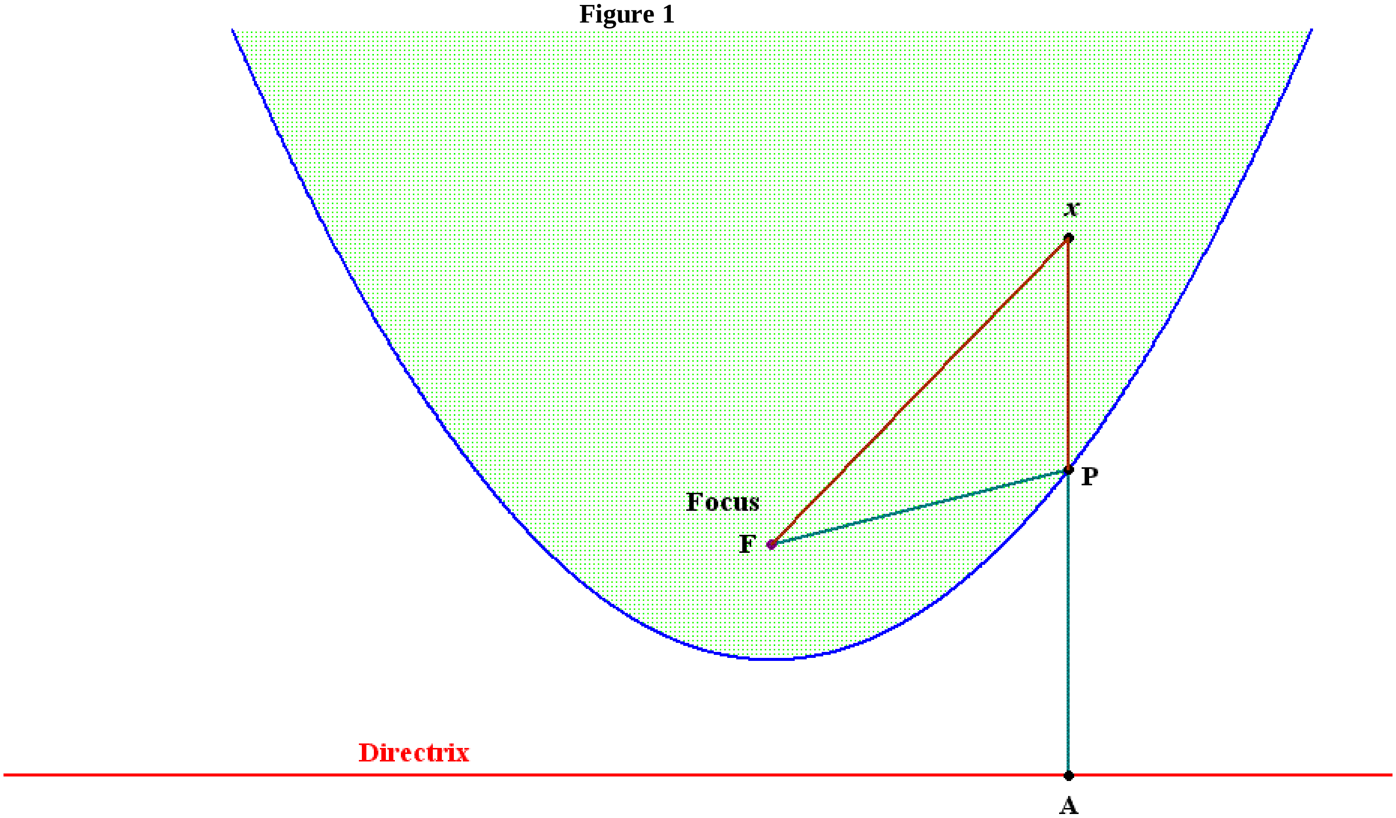
Construction of a Tangent Line to a Parabola

Consider a parabola with Focus at point **F** and Directrix **D**.
Lemma 1: A point ***x*** lies within the envelope of the parabola if and only if **F*x* < A*x***, where **A** is the intersection of **D** and a line perpendicular to **D** passing through ***x***.

Figure 1

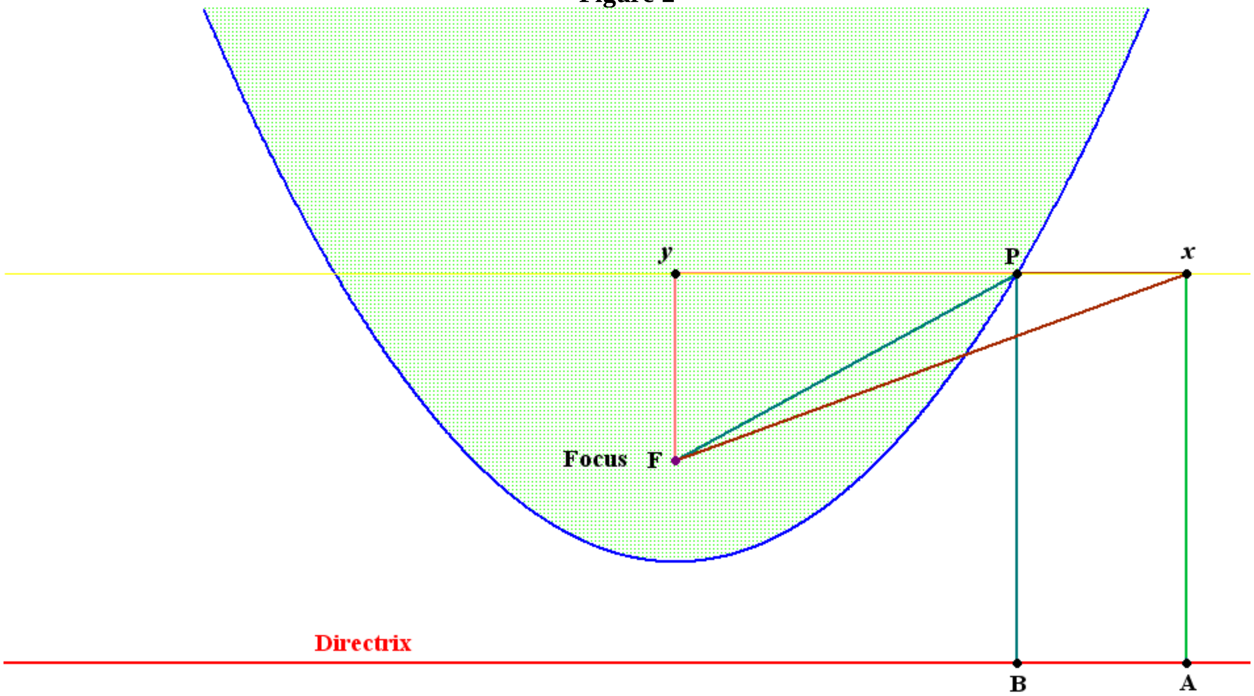


Proof

Sufficiency: As illustrated in Figure 1, pick a point *x* from within the envelope of the parabola. Construct ***x*A** by dropping a perpendicular to **D** through ***x***. Let **P** be the point where the parabola intersects ***x*A** . By the triangle inequality,
F*x* < **FP** + **P*x***. But **FP** = **AP**, so **F*x*** < **AP** + **P*x*** = **A*x***.

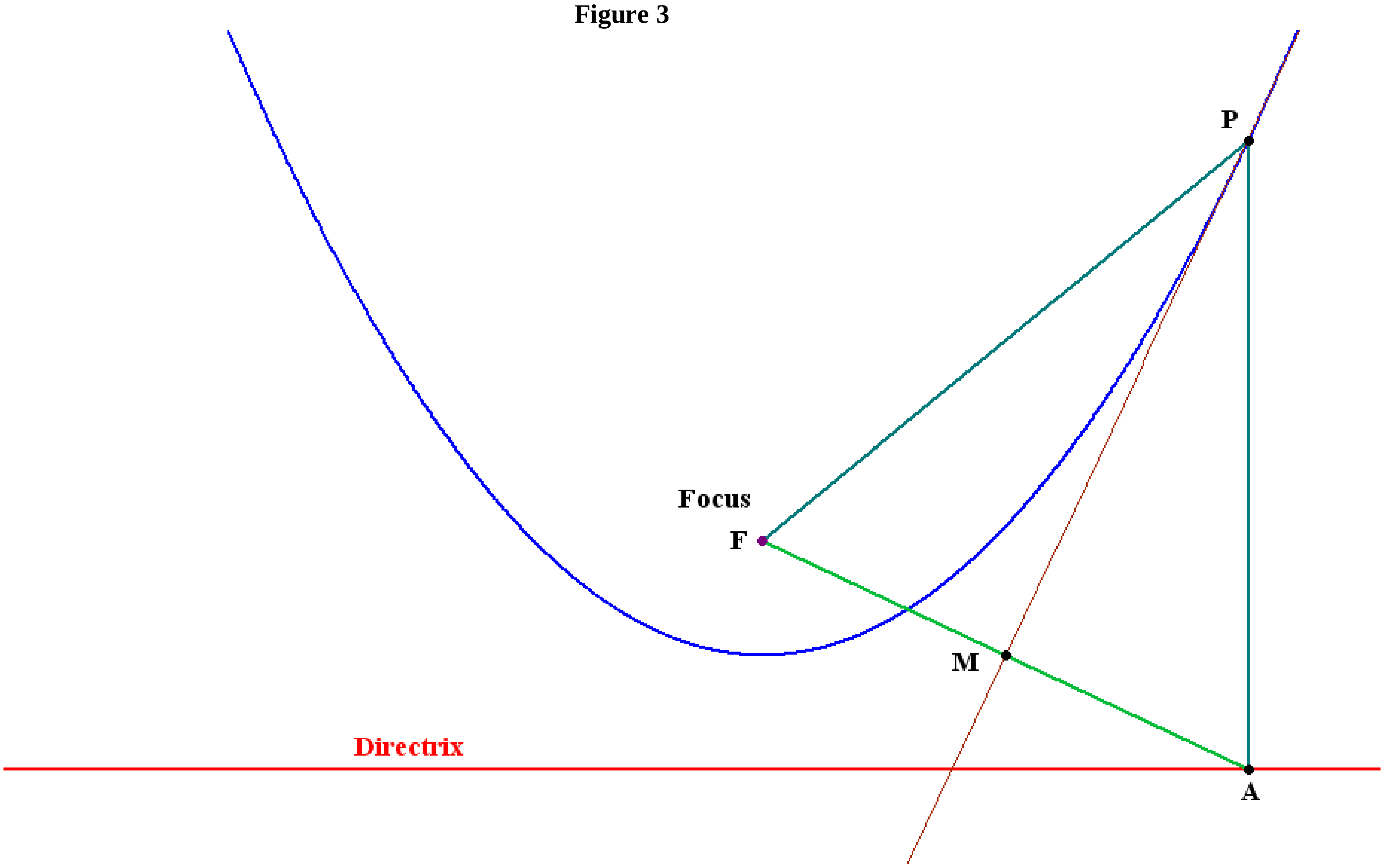
Necessity: As illustrated in Figure 2, pick a point *x* from outside the envelope of the parabola. Construct ***x*A** by dropping a perpendicular to **D** through ***x***. Through ***x*** construct line **C** parallel to **D**. Let **P** be the point where the parabola intersects **C**. Let ***y*** be the intersection of **C** with a line through **F** perpendicular to **C**. Let **B** be the intersection of **D** with a line through **P** perpendicular to **D**. So **A*x*** = **BP**. By the Pythagorean Theorem $(\mathrm{F}x)^2 = (\mathrm{F}y)^2 + (yx)^2 > (\mathrm{F}y)^2 + (y\mathrm{P})^2 = (\mathrm{FP})^2 = (\mathrm{BP})^2$. So **F*x*** > **A*x***. So by contraposition, if **F*x*** < **A*x***, *x* must lie inside the envelope of the parabola.

Figure 2



Construction of the Tangent Line to the Parabola at a point P:
As illustrated in Figure 3, let **A** be the intersection of **D** with a line through **P** perpendicular to **D**. Then **FP** = **PA** and triangle **FPA** is isosceles. Let **M** be the midpoint of segment **FA** then line **PM** is the perpendicular bisector. This line is the tangent line to the parabola at **P**.

Figure 3



Proof:
Pick any point *x* on the perpendicular bisector of **FA**. By SAS, triangle **FM*x*** is congruent to triangle **AM*x*** and so **F*x*** = **A*x*** . Let **B** be the intersection of **D** with a line through ***x*** perpendicular to **D**. If ***x*** is different than **P**, **B*x*A** is a right triangle with hypotenuse **A*x*** so **A*x*** > ***x*B** . Thus **F*x*** > ***x*B** and by the lemma point ***x*** must lie outside the envelope of the parabola. This is illustrated in figures 4 and 5. Thus the only point where the perpendicular bisector of **FA** intersects the parabola is **P**.

Figure 4

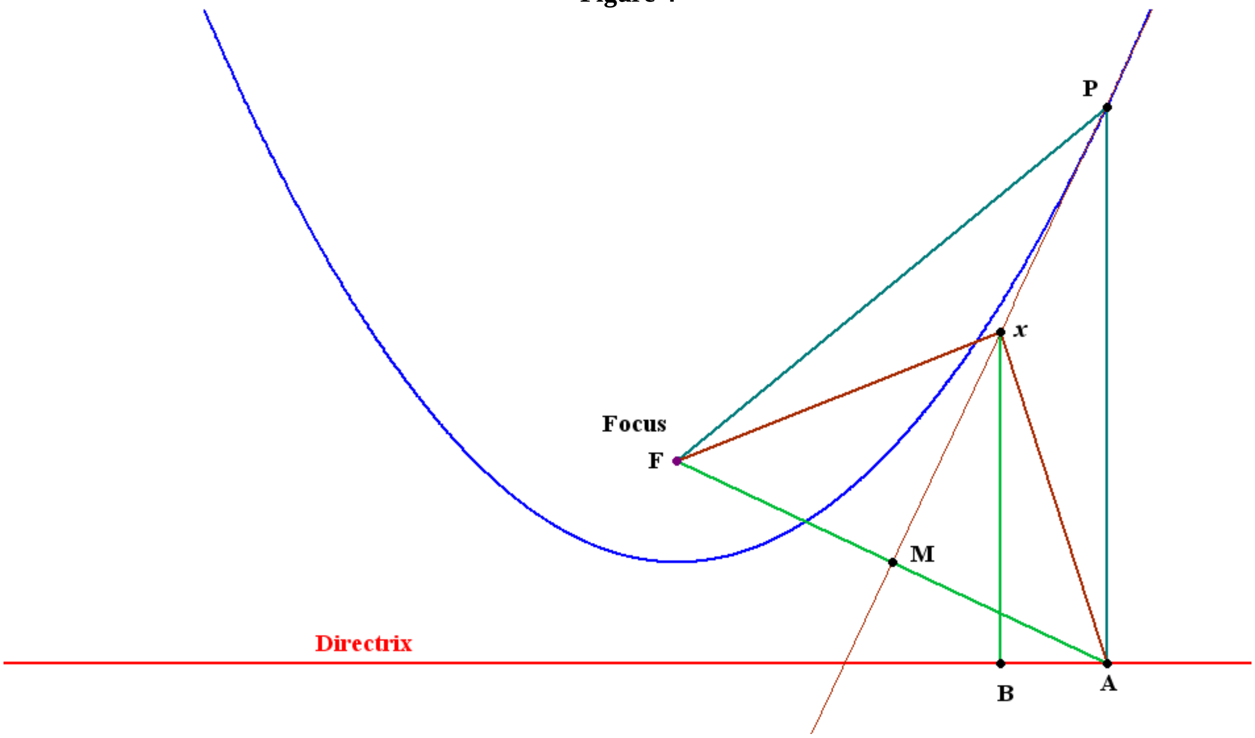
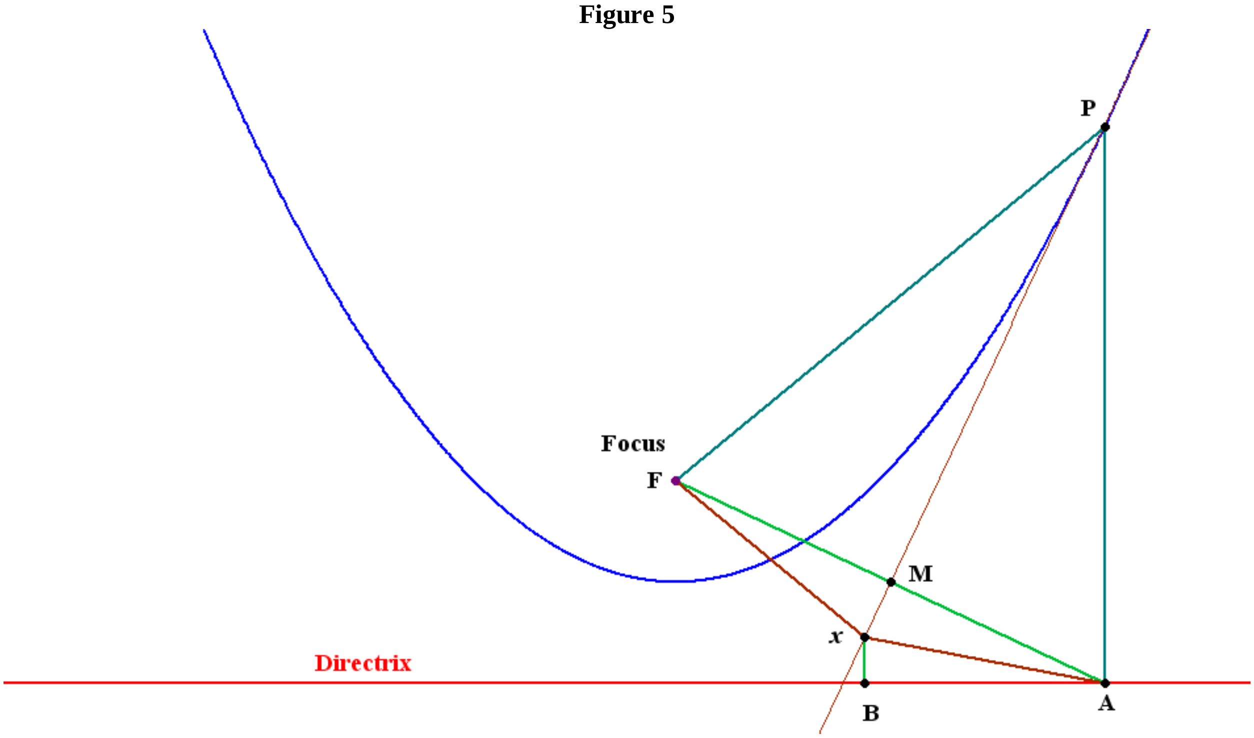


Figure 5



This construction also provides a nice proof of the reflective property of a parabola. As illustrated in Figure 6 consider a light ray travelling from **I** parallel to the axis of the parabola, i.e. perpendicular to the directrix. The light ray intersects the parabola at **P** with a normal **PN** parallel to segment **FA**. The "angle of incidence", $\alpha = \measuredangle \mathrm{IPN}$, must equal the angle of reflection.

Since $\measuredangle \mathrm{IPN} = \frac{\pi}{2} - \measuredangle \mathrm{MPA} = \frac{\pi}{2} - \measuredangle \mathrm{MPF} = \measuredangle \mathrm{NPF}$, the reflected ray must pass through the focal point **F**. An essentially identical argument shows that a light source located at **F** has its rays reflected in a beam parallel to the axis of the parabola.

Figure 6