

# The Logistic Equation with Harvesting

## Introduction:

For positive  $k$ ,  $L$  and  $R$  the logistic differential equation with constant “harvesting” is given by

$$\frac{dN}{dt} = f(N, k, L, R) = kN \left( 1 - \frac{N}{L} \right) - R. \quad (1)$$

Here  $N$  is the population of a species at time  $t$ ,  $k$  is a rate of growth constant,  $L$  is the limiting population in the absence of harvesting, and  $R$  is the harvesting rate, i.e., how many individuals are removed per unit time. The initial condition is that  $N(0) = N_0$ .  $N$  and  $L$  have units of population,  $k$  has units of  $\text{time}^{-1}$ , and  $R$  has units of population divided by time. Just based on units, the dimensionless ratios  $\frac{R}{kL}$  and  $\frac{R}{kN_0}$  must be the quantities which characterize the impact of harvesting on the qualitative behavior of the solution of equation (1).

The problem can be solved by analytic methods for all values of  $k$ ,  $L$ , and  $R$ . However, the equation also illustrates the utility of the qualitative approach where one studies the equilibrium solutions determined by the zeroes of  $f(N, k, L, R)$ . The model has enough structure to produce both non-trivial and interesting solutions. As such it provides a nice explicit example of many of the ideas students encounter in an elementary ODE’s course. In addition, the analysis of the solution as the parameters approach critical values provides a nice reinforcement to the calculation of limits and Maclaurin series.

## Analysis:

### 1. The limit as $L \rightarrow \infty$ :

As  $L \rightarrow \infty$  the equation (1) approaches the linear differential equation

$$\frac{dN}{dt} = kN - R. \quad (2)$$

This can be solved either by separation of variables or the use of an integrating factor. Using the latter approach, one obtains

$$\begin{aligned} \frac{dN}{dt} e^{-kt} - kN e^{-kt} &= -R e^{-kt} \\ \frac{d}{dt} (N e^{-kt}) &= \frac{d}{dt} \left( \frac{R}{k} e^{-kt} \right) \end{aligned}$$

Hence by integration,  $N(t) = \frac{R}{k} + C e^{kt}$  for some constant  $C$ . Using the initial condition,  $C = N_0 - \frac{R}{k}$ . So the solution of equation (2) can be written as

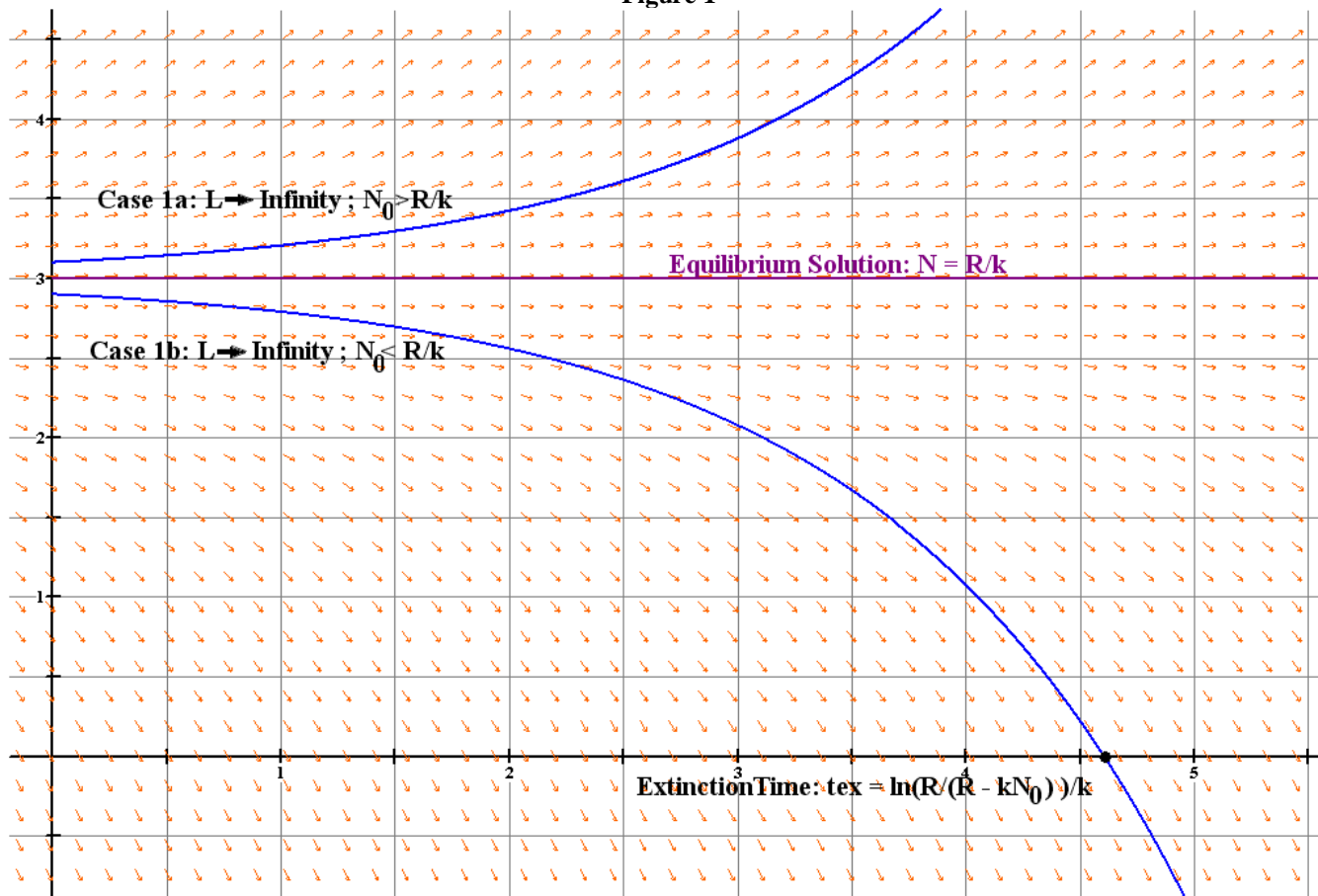
$$N(t) = \frac{R}{k} + \left( N_0 - \frac{R}{k} \right) e^{kt}. \quad (3)$$

Now the problem demands non-negative values of  $N$ , hence the species becomes extinct if  $N_0 < \frac{R}{k}$  with an extinction time

given by  $t_{ex} = \frac{1}{k} \ln \left( \frac{R}{R - kN_0} \right)$ . If  $N_0 > \frac{R}{k}$  the population grows exponentially since there is no limiting population. Thus

the equilibrium solution  $N(t) = N_0 = \frac{R}{k}$  is unstable since small departures from  $\frac{R}{k}$  in the initial population cause the solution to rapidly depart from the equilibrium value. This is illustrated in Figure 1.

Figure 1



## 2. Solutions for positive $L$ :

The equilibrium or fixed point solutions of equation (1) are the values of  $N$  for which

$$f(N, k, L, R) = kN \left( 1 - \frac{N}{L} \right) - R = -\frac{k}{L} \left[ \left( N - \frac{L}{2} \right)^2 - \left( \frac{L^2}{4} - \frac{RL}{k} \right) \right] = 0. \text{ There are two equilibrium solutions:}$$

$$N = \frac{L}{2} \left( 1 \pm \sqrt{1 - \frac{4R}{kL}} \right). \text{ These are real if and only if } R \leq \frac{kL}{4}.$$

**Case 2a:**  $R < \frac{kL}{4}$ :

The two equilibrium solutions are  $N_\ell = \frac{L}{2} \left( 1 - \sqrt{1 - \frac{4R}{kL}} \right)$  and  $N_u = \frac{L}{2} \left( 1 + \sqrt{1 - \frac{4R}{kL}} \right)$ . Consider

$$f(N_u + \varepsilon, k, L, R) = -\frac{k}{L} \left[ \left( N_u + \varepsilon - \frac{L}{2} \right)^2 - \left( \frac{L^2}{4} - \frac{RL}{k} \right) \right] = -\frac{k\varepsilon}{L} \left[ L \sqrt{1 - \frac{4R}{kL}} + \varepsilon \right]. \text{ Substituting } N = N_u + \varepsilon \text{ into equation (1)}$$

yields  $\frac{d\varepsilon}{dt} = -k\varepsilon \left( \sqrt{1 - \frac{4R}{kL}} + \frac{\varepsilon}{L} \right)$ . For small departures from  $N_u$ , the solutions behave like  $\varepsilon(t) \approx \varepsilon_0 \text{EXP} \left[ - \left( k \sqrt{1 - \frac{4R}{kL}} \right) t \right]$ .

Thus  $N(t) = N_u = \frac{L}{2} \left( 1 + \sqrt{1 - \frac{4R}{kL}} \right)$  is a stable equilibrium solution. Similarly,

$$f(N_\ell + \varepsilon, k, L, R) = -\frac{k}{L} \left[ \left( N_\ell + \varepsilon - \frac{L}{2} \right)^2 - \left( \frac{L^2}{4} - \frac{RL}{k} \right) \right] = \frac{k\varepsilon}{L} \left[ L\sqrt{1 - \frac{4R}{kL}} - \varepsilon \right].$$

Substituting  $N = N_\ell + \varepsilon$  into equation (1)

yields  $\frac{d\varepsilon}{dt} = k\varepsilon \left( \sqrt{1 - \frac{4R}{kL}} - \frac{\varepsilon}{L} \right)$ . For small departures from  $N_\ell$ , the solutions initially grow exponentially like

$$\varepsilon(t) \approx \varepsilon_0 \text{EXP} \left[ \left( k\sqrt{1 - \frac{4R}{kL}} \right) t \right].$$

Thus  $N(t) = N_\ell = \frac{L}{2} \left( 1 - \sqrt{1 - \frac{4R}{kL}} \right)$  is an unstable equilibrium solution.

The analytic solution of equation (1) when  $R < \frac{kL}{4}$  is accomplished by separation of variables.

$$\frac{dN}{\left( N - \frac{L}{2} \right)^2 - \frac{L^2}{4} \left( 1 - \frac{4R}{kL} \right)} = -\frac{k}{L} dt \quad (4)$$

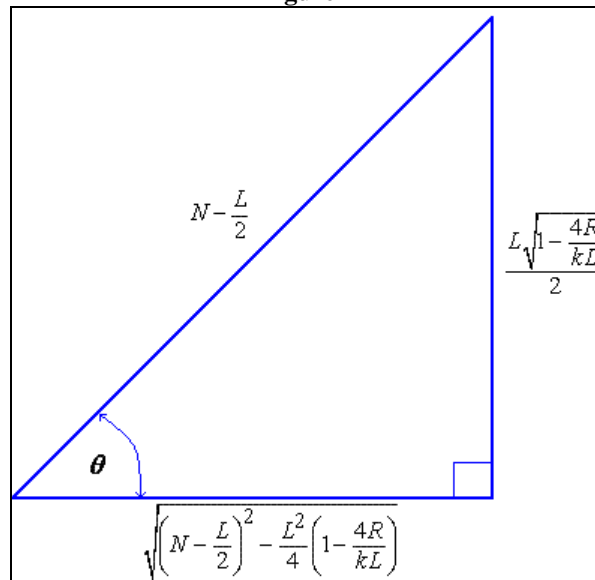
**Case 2a i:**  $N_0 > N_u = \frac{L}{2} \left( 1 + \sqrt{1 - \frac{4R}{kL}} \right)$

Since  $\left( N_0 - \frac{L}{2} \right)^2 - \frac{L^2}{4} \left( 1 - \frac{4R}{kL} \right) > 0$ ,  $N'(0) < 0$  and  $N(t)$  decreases. A turning point requires that  $N'(t) = 0$  which requires that  $N(t) = N_u$  or  $N(t) = N_\ell$ . A solution can not pass through a fixed point, so  $N(t)$  has no turning points. Hence  $N'(t) < 0$  for all  $t$ , but  $N(t)$  must have a lower bound since it can not cross  $N_u$ . Thus for all  $t$ ,  $N(t)$  is a decreasing continuous function with  $N(t) > N_u$ . Therefore the  $\lim_{t \rightarrow \infty} N(t)$  must exist. But this implies that  $\lim_{t \rightarrow \infty} N'(t) = 0$  and hence  $\lim_{t \rightarrow \infty} N(t)$

must equal the equilibrium solution  $N_u$ . Since  $N(t) > N_u$ ,  $\left( N(t) - \frac{L}{2} \right)^2 - \frac{L^2}{4} \left( 1 - \frac{4R}{kL} \right) > 0$ ; this motivates the

trigonometric substitution  $N = \frac{L}{2} + \csc(\theta) \frac{L\sqrt{1 - \frac{4R}{kL}}}{2}$ .

Figure 2



$$dN = -\frac{L\sqrt{1-\frac{4R}{kL}}}{2} \csc(\theta) \cot(\theta) d\theta \text{ and } \left(N(t) - \frac{L}{2}\right)^2 - \frac{L^2}{4} \left(1 - \frac{4R}{kL}\right) = \frac{L^2}{4} \left(1 - \frac{4R}{kL}\right) \cot^2(\theta). \text{ Equation (4) then becomes}$$

$$\frac{\csc(\theta) d\theta}{\frac{L}{2}\sqrt{1-\frac{4R}{kL}} \cot(\theta)} = \frac{\sec(\theta) d\theta}{\frac{L}{2}\sqrt{1-\frac{4R}{kL}}} = \frac{k}{L} dt \text{ which integrates as } \ln[\sec(\theta) + \tan(\theta)] = \frac{k}{2}\sqrt{1-\frac{4R}{kL}} t + \frac{\ln(A)}{2}, \text{ where } \frac{\ln(A)}{2} \text{ is}$$

the constant of integration. Eliminating the natural logarithm gives the following result:

$$\frac{N - \frac{L}{2} + \frac{L}{2}\sqrt{1-\frac{4R}{kL}}}{\sqrt{\left(N - \frac{L}{2}\right)^2 - \frac{L^2}{4} \left(1 - \frac{4R}{kL}\right)}} = \sqrt{A} \exp\left(\frac{k}{2}\sqrt{1-\frac{4R}{kL}} t\right).$$

Squaring both sides gives

$$\frac{\left(N - \frac{L}{2} + \frac{L}{2}\sqrt{1-\frac{4R}{kL}}\right)^2}{\left(N - \frac{L}{2}\right)^2 - \frac{L^2}{4} \left(1 - \frac{4R}{kL}\right)} = \frac{(N - N_\ell)^2}{\left(N - \frac{L}{2} - \frac{L}{2}\sqrt{1-\frac{4R}{kL}}\right)\left(N - \frac{L}{2} + \frac{L}{2}\sqrt{1-\frac{4R}{kL}}\right)} = \frac{N - N_\ell}{N - N_u} = A \exp\left(k\sqrt{1-\frac{4R}{kL}} t\right).$$

The initial condition at  $t = 0$  requires that  $A = \frac{N_0 - N_\ell}{N_0 - N_u}$ . The following sequence of steps solves for  $N(t)$ .

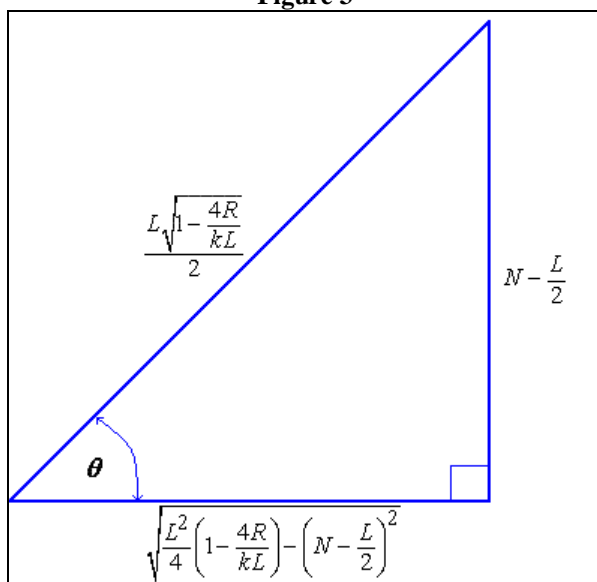
$$\begin{aligned} (N(t) - N_u) \left[ A \exp\left(k\sqrt{1-\frac{4R}{kL}} t\right) \right] &= N(t) - N_\ell \\ N(t) \left[ A \exp\left(k\sqrt{1-\frac{4R}{kL}} t\right) - 1 \right] &= N_u A \exp\left(k\sqrt{1-\frac{4R}{kL}} t\right) - N_\ell \\ N(t) &= \frac{N_u A \exp\left(k\sqrt{1-\frac{4R}{kL}} t\right) - N_\ell}{A \exp\left(k\sqrt{1-\frac{4R}{kL}} t\right) - 1} = \frac{N_u A \exp\left(k\sqrt{1-\frac{4R}{kL}} t\right) - N_u}{A \exp\left(k\sqrt{1-\frac{4R}{kL}} t\right) - 1} + \frac{N_u - N_\ell}{A \exp\left(k\sqrt{1-\frac{4R}{kL}} t\right) - 1} \\ &= N_u + \frac{N_u - N_\ell}{A \exp\left(k\sqrt{1-\frac{4R}{kL}} t\right) - 1} = N_u + \frac{N_u - N_\ell}{\frac{N_0 - N_\ell}{N_0 - N_u} \exp\left(k\sqrt{1-\frac{4R}{kL}} t\right) - 1} \\ N(t) &= N_u + \frac{(N_u - N_\ell)(N_0 - N_u)}{(N_0 - N_\ell) \exp\left(k\sqrt{1-\frac{4R}{kL}} t\right) + N_u - N_0} \quad (5) \end{aligned}$$

The solution expressed in equation (5) confirms that  $\lim_{t \rightarrow \infty} N(t) = N_u$ .

**Case 2a ii:**  $\frac{L}{2} \left(1 - \sqrt{1 - \frac{4R}{kL}}\right) = N_\ell < N_0 < N_u = \frac{L}{2} \left(1 + \sqrt{1 - \frac{4R}{kL}}\right)$

Since  $-\frac{L}{2}\sqrt{1-\frac{4R}{kL}} < N_0 - \frac{L}{2} < \frac{L}{2}\sqrt{1-\frac{4R}{kL}}$ ,  $\left(N_0 - \frac{L}{2}\right)^2 - \frac{L^2}{4}\left(1 - \frac{4R}{kL}\right) < 0$ , so  $N'(0) > 0$  and  $N(t)$  increases. Again a turning point requires that  $N(t) = N_u$  or  $N(t) = N_\ell$ , so  $N(t)$  has no turning points. Hence  $N'(t) > 0$  for all  $t$ , but  $N(t)$  must have an upper bound since it can not cross  $N_u$ . Thus, for all  $t$ ,  $N(t)$  is an increasing continuous function with  $N_\ell < N(t) < N_u$ . Therefore the  $\lim_{t \rightarrow \infty} N(t)$  must exist. Again this implies that  $\lim_{t \rightarrow \infty} N'(t) = 0$  and hence  $\lim_{t \rightarrow \infty} N(t)$  must equal the equilibrium solution  $N_u$ . Since  $-\frac{L}{2}\sqrt{1-\frac{4R}{kL}} < N(t) - \frac{L}{2} < \frac{L}{2}\sqrt{1-\frac{4R}{kL}}$ ,  $\frac{L^2}{4}\left(1 - \frac{4R}{kL}\right) - \left(N(t) - \frac{L}{2}\right)^2 > 0$ ; this motivates the trigonometric substitution  $N = \frac{L}{2} + \frac{L\sqrt{1-\frac{4R}{kL}}}{2}\sin(\theta)$ .

Figure 3



$$dN = \frac{L\sqrt{1-\frac{4R}{kL}}}{2}\cos(\theta)d\theta \text{ and } \left(N(t) - \frac{L}{2}\right)^2 - \frac{L^2}{4}\left(1 - \frac{4R}{kL}\right) = -\frac{L^2}{4}\left(1 - \frac{4R}{kL}\right)\cos^2(\theta). \text{ Equation (4) then becomes}$$

$$\frac{\sec(\theta)d\theta}{\frac{L\sqrt{1-\frac{4R}{kL}}}{2}} = \frac{k}{L}dt \text{ which is identical to the result of Case 2.a i when } N_0 > N_u. \text{ Thus the solution again takes the form of}$$

equation (5) which in turn confirms that when  $N_\ell < N_0 < N_u$ , the  $\lim_{t \rightarrow \infty} N(t) = N_u$ . It should be noted that in the absence of

harvesting, i.e.,  $R = 0$ , that  $N_u = \frac{L}{2}\left(1 + \sqrt{1 - \frac{4R}{kL}}\right)$  becomes  $L$  and  $N_\ell = \frac{L}{2}\left(1 - \sqrt{1 - \frac{4R}{kL}}\right)$  vanishes. Equation (5) then reduce to the familiar solution of the logistic model,

$$N(t) = L + \frac{L(N_0 - L)}{N_0 \exp(kt) + L - N_0} = \frac{N_0 L \exp(kt)}{N_0 \exp(kt) + L - N_0}.$$

If  $R < \frac{kL}{4}$  and  $N_0 > N_\ell$  the effect of harvesting is to reduce the limiting population from  $L$  to  $N_u = L - \frac{L}{2}\left(1 - \sqrt{1 - \frac{4R}{kL}}\right)$ .

For  $|x| < 1$ , an application of the binomial expansion gives the following result.

$$\begin{aligned}\sqrt{1-x} &= 1 - \frac{1}{2}x - \frac{1}{2^2 2!}x^2 - \frac{1 \cdot 3}{2^3 3!}x^3 - \frac{1 \cdot 3 \cdot 5}{2^4 4!}x^4 - \frac{1 \cdot 3 \cdot 5 \cdot 7}{2^5 5!}x^5 \dots \\ &= 1 - \frac{1}{2}x - \frac{1}{2^2 2!}x^2 - \frac{1 \cdot 2(1) \cdot 3}{2^3 2(1)!3!}x^3 - \frac{1 \cdot 2(1) \cdot 3 \cdot 2(2) \cdot 5}{2^4 2^2 2!4!}x^4 - \frac{1 \cdot 2(1) \cdot 3 \cdot 2(2) \cdot 5 \cdot 2(3) \cdot 7}{2^5 2^3 3!5!}x^5 \dots \\ &= 1 - \frac{1}{2}x - \frac{1}{2^2 2!}x^2 - \frac{3!}{2^3 2(1)!3!}x^3 - \frac{5!}{2^4 2^2 2!4!}x^4 - \frac{7!}{2^5 2^3 3!5!}x^5 \dots \\ &= 1 - \frac{1}{2}x - \sum_{n=2}^{\infty} \frac{(2n-3)!}{2^{2(n-1)}(n-2)!n!}x^n\end{aligned}$$

$$\begin{aligned}\text{So } N_\ell &= \frac{L}{2} \left( 1 - \sqrt{1 - \frac{4R}{kL}} \right) = \frac{L}{2} \frac{2R}{kL} + \frac{L}{2} \sum_{n=2}^{\infty} \frac{(2n-3)! 2^{2n} R^n}{2^{2(n-1)} n!(n-2)!k^n L^n} = \frac{R}{k} + 2 \sum_{n=2}^{\infty} \frac{(2n-3)! R^n}{n!(n-2)!k^n L^{n-1}} \text{ and} \\ N_u &= \frac{L}{2} \left( 1 + \sqrt{1 - \frac{4R}{kL}} \right) = L - \frac{L}{2} \frac{2R}{kL} - \frac{L}{2} \sum_{n=2}^{\infty} \frac{(2n-3)! 2^{2n} R^n}{2^{2(n-1)} n!(n-2)!k^n L^n} = L - \frac{R}{k} - 2 \sum_{n=2}^{\infty} \frac{(2n-3)! R^n}{n!(n-2)!k^n L^{n-1}}.\end{aligned}$$

As  $L$  becomes very large the stable equilibrium solution at  $N_u$  moves to infinity and the unstable equilibrium solution at  $N_\ell$  becomes  $\frac{R}{k}$ . If  $\frac{R}{kL}$  is small and  $N_0 > N_\ell$  harvesting reduces the limiting population by an amount approximately equal to  $\frac{R}{k}$ . Since for large  $L$ ,  $N_u$  approaches  $L$ , the limit of equation (5) can be evaluated as follows.

$$\begin{aligned}\lim_{L \rightarrow \infty} N(t) &= \lim_{L \rightarrow \infty} N_u + \frac{(N_u - N_\ell)(N_0 - N_u)}{(N_0 - N_\ell) \exp\left(k\sqrt{1 - \frac{4R}{kL}}t\right) + N_u - N_0} = \lim_{L \rightarrow \infty} L + \frac{L\left(L - \frac{R}{k}\right)\left(\frac{N_0}{L} - 1\right)}{\left(N_0 - \frac{R}{k}\right) \exp\left(k\sqrt{1 - \frac{4R}{kL}}t\right) + L - N_0} \\ &= \lim_{L \rightarrow \infty} \frac{L^2 - LN_0 + L\left(N_0 - \frac{R}{k}\right) \exp\left(k\sqrt{1 - \frac{4R}{kL}}t\right) + L\left(N_0 - L - \frac{RN_0}{kL} + \frac{R}{k}\right)}{\left(N_0 - \frac{R}{k}\right) \exp\left(k\sqrt{1 - \frac{4R}{kL}}t\right) + L - N_0} \\ &= \lim_{L \rightarrow \infty} \frac{L}{L} \cdot \frac{\left(N_0 - \frac{R}{k}\right) \exp\left(k\sqrt{1 - \frac{4R}{kL}}t\right) + \left(-\frac{RN_0}{kL} + \frac{R}{k}\right)}{\frac{1}{L}\left(N_0 - \frac{R}{k}\right) \exp\left(k\sqrt{1 - \frac{4R}{kL}}t\right) + 1 - \frac{N_0}{L}} = \frac{R}{k} + \left(N_0 - \frac{R}{k}\right)e^{kt}\end{aligned}$$

This agrees with equation (3), the solution of equation (1) in the limit that  $L$  goes to infinity.

If  $R$  is negative (stocking rather than harvesting the population)  $1 - \frac{4R}{kL} > 1$  and  $N_\ell < 0$ . For negative  $R$  an analysis essentially the same as that just presented demonstrates that equation (5) is the solution to equation (1) for any initial population. Solutions with  $0 < N_0 < N_u$  monotonically increase to the equilibrium solution  $N_u$ , while solutions with  $N_0 > N_u$  monotonically decrease to  $N_u$ . Extinction never occurs. If  $\frac{-R}{kL}$  is small the effect of stocking is to increase the

limiting population by an amount approximately equal to  $\frac{-R}{k}$ .

**Case 2a iii:**  $N_0 < N_\ell = \frac{L}{2} \left( 1 - \sqrt{1 - \frac{4R}{kL}} \right)$

Since  $N_0 - \frac{L}{2} < -\frac{L}{2} \sqrt{1 - \frac{4R}{kL}}$ ,  $\left( N_0 - \frac{L}{2} \right)^2 - \frac{L^2}{4} \left( 1 - \frac{4R}{kL} \right) > 0$ , so  $N'(0) < 0$  and  $N(t)$  decreases. As in the previous two cases,  $N(t)$  has no turning points. Since  $N(t)$  is initially below both equilibrium solutions it must decrease without a lower

limit. Thus, if  $N_0 < N_\ell$ , the population always becomes extinct.  $N_0 < \frac{L}{2}$  and  $\left( \frac{L}{2} - N_0 \right)^2 - \frac{L^2}{4} \left( 1 - \frac{4R}{kL} \right) > 0$ . As

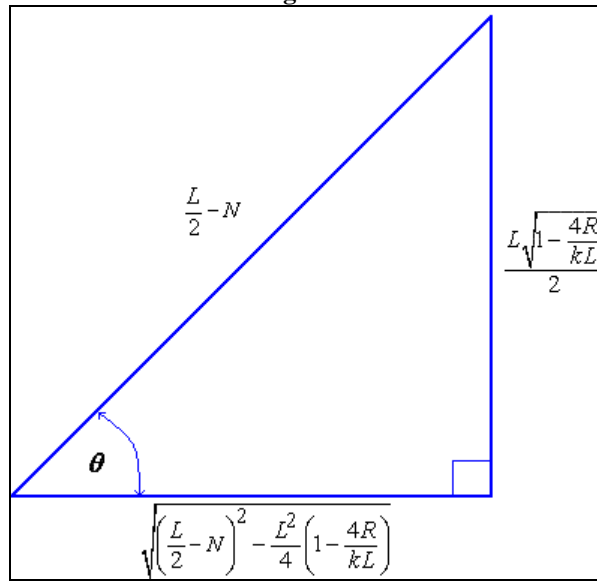
$N(t)$  decreases from  $N_0$ ,  $N(t) - \frac{L}{2} < -\frac{L}{2} \sqrt{1 - \frac{4R}{kL}}$  so  $\left( N(t) - \frac{L}{2} \right)^2 - \frac{L^2}{4} \left( 1 - \frac{4R}{kL} \right) > 0$  in this domain. This motivates the

trigonometric substitution  $N = \frac{L}{2} - \frac{L \sqrt{1 - \frac{4R}{kL}}}{2} \csc(\theta)$ , so  $dN = -\frac{L \sqrt{1 - \frac{4R}{kL}}}{2} \csc(\theta) \cot(\theta) d\theta$  and

$$\left( \frac{L}{2} - N(t) \right)^2 - \frac{L^2}{4} \left( 1 - \frac{4R}{kL} \right) = \frac{L^2}{4} \left( 1 - \frac{4R}{kL} \right) \cot^2(\theta). \text{ Equation (4) then becomes}$$

$$\frac{\csc(\theta) d\theta}{\frac{L}{2} \sqrt{1 - \frac{4R}{kL}} \cot(\theta)} = \frac{\sec(\theta) d\theta}{\frac{L}{2} \sqrt{1 - \frac{4R}{kL}}} = -\frac{k}{L} dt \text{ which integrates as } \ln[\sec(\theta) + \tan(\theta)] = -\frac{k}{2} \sqrt{1 - \frac{4R}{kL}} t + \frac{\ln(A)}{2}.$$

**Figure 4**



Eliminating the natural logarithm gives the following result:  $\frac{\frac{L}{2} - N + \frac{L}{2} \sqrt{1 - \frac{4R}{kL}}}{\sqrt{\left( \frac{L}{2} - N \right)^2 - \frac{L^2}{4} \left( 1 - \frac{4R}{kL} \right)}} = \sqrt{A} \exp\left( -\frac{k}{2} \sqrt{1 - \frac{4R}{kL}} t \right).$

Squaring both sides, gives

$$\frac{\left( \frac{L}{2} + \frac{L}{2} \sqrt{1 - \frac{4R}{kL}} - N \right)^2}{\left( \frac{L}{2} - N \right)^2 - \frac{L^2}{4} \left( 1 - \frac{4R}{kL} \right)} = \frac{(N_u - N)^2}{\left( \frac{L}{2} - \frac{L}{2} \sqrt{1 - \frac{4R}{kL}} - N \right) \left( \frac{L}{2} + \frac{L}{2} \sqrt{1 - \frac{4R}{kL}} - N \right)} = \frac{N_u - N}{N_\ell - N} = A \exp\left( -k \sqrt{1 - \frac{4R}{kL}} t \right).$$

The initial condition at  $t = 0$  requires that  $A = \frac{N_u - N_0}{N_\ell - N_0}$ . The following sequence of steps solves for  $N(t)$ .

$$\begin{aligned}
 N(t) \left[ 1 - A \exp \left( -k \sqrt{1 - \frac{4R}{kL}} t \right) \right] &= N_u - N_\ell A \exp \left( -k \sqrt{1 - \frac{4R}{kL}} t \right) \\
 N(t) &= \frac{N_u - N_\ell A \exp \left( -k \sqrt{1 - \frac{4R}{kL}} t \right)}{1 - A \exp \left( -k \sqrt{1 - \frac{4R}{kL}} t \right)} = \frac{N_u - N_u A \exp \left( -k \sqrt{1 - \frac{4R}{kL}} t \right)}{1 - A \exp \left( -k \sqrt{1 - \frac{4R}{kL}} t \right)} + \frac{(N_u - N_\ell) A \exp \left( -k \sqrt{1 - \frac{4R}{kL}} t \right)}{1 - A \exp \left( -k \sqrt{1 - \frac{4R}{kL}} t \right)} \\
 &= N_u + \frac{A(N_u - N_\ell)}{\exp \left( k \sqrt{1 - \frac{4R}{kL}} t \right) - A} = N_u + \frac{(N_u - N_\ell) \frac{N_u - N_0}{N_\ell - N_0}}{\exp \left( k \sqrt{1 - \frac{4R}{kL}} t \right) - \frac{N_u - N_0}{N_\ell - N_0}} = N_u + \frac{(N_u - N_\ell)(N_u - N_0)}{(N_\ell - N_0) \exp \left( k \sqrt{1 - \frac{4R}{kL}} t \right) - (N_u - N_0)} \\
 &= N_u + \frac{(-1)(N_u - N_\ell)(N_u - N_0)}{(N_0 - N_\ell) \exp \left( k \sqrt{1 - \frac{4R}{kL}} t \right) + N_u - N_0} = N_u + \frac{(N_u - N_\ell)(N_0 - N_u)}{(N_0 - N_\ell) \exp \left( k \sqrt{1 - \frac{4R}{kL}} t \right) + N_u - N_0}
 \end{aligned}$$

This last expression is identical to equation (5). Furthermore, in equation (5), if  $N_0 = N_u$ ,  $N(t) = N_u$  and if  $N_0 = N_\ell$ ,

$N(t) = N_\ell$ . Thus, it has been established that if  $R < \frac{kL}{4}$ , the solution of equation (1) for any initial condition is given by the

one formula. 
$$N(t) = N_u + \frac{(N_u - N_\ell)(N_0 - N_u)}{(N_0 - N_\ell) \exp \left( k \sqrt{1 - \frac{4R}{kL}} t \right) + N_u - N_0}.$$

For  $N_0 < N_\ell$  it is still the case that  $\lim_{t \rightarrow \infty} N(t) = N_u$ ; however, the solution shows very different behavior than the previous

two cases with  $N_0 > N_\ell$ . Since  $0 < N_\ell - N_0 < N_u - N_0$  the denominator  $(N_0 - N_\ell) \exp \left( k \sqrt{1 - \frac{4R}{kL}} t \right) + N_u - N_0$  is initially

$N_u - N_\ell$  which is positive. The negative term  $(N_0 - N_\ell) \exp \left( k \sqrt{1 - \frac{4R}{kL}} t \right)$  causes the denominator to vanish at the critical

value of  $t$  given by  $t_c = \frac{1}{k \sqrt{1 - \frac{4R}{kL}}} \ln \left( \frac{N_u - N_0}{N_\ell - N_0} \right)$ . There is a vertical asymptote in the graph at  $t = t_c$ . As  $t \rightarrow t_c^-$ ,

$N(t) \rightarrow -\infty$ . So for exactly one value of  $t$ ,  $t_{ex}$ , with  $0 < t_{ex} < t_c$ ,  $N(t_{ex}) = 0$ . Rewriting equation (5) as a single fraction,

$$\begin{aligned}
 N(t) &= N_u + \frac{(N_u - N_\ell)(N_0 - N_u)}{(N_0 - N_\ell) \exp \left( k \sqrt{1 - \frac{4R}{kL}} t \right) + N_u - N_0} = \frac{N_u (N_0 - N_\ell) \exp \left( k \sqrt{1 - \frac{4R}{kL}} t \right) + N_u (N_u - N_0) + (N_u - N_\ell)(N_0 - N_u)}{(N_0 - N_\ell) \exp \left( k \sqrt{1 - \frac{4R}{kL}} t \right) + N_u - N_0} \\
 &= \frac{N_\ell (N_u - N_0) - N_u (N_\ell - N_0) \exp \left( k \sqrt{1 - \frac{4R}{kL}} t \right)}{(N_0 - N_\ell) \exp \left( k \sqrt{1 - \frac{4R}{kL}} t \right) + N_u - N_0}
 \end{aligned}$$

Thus, for  $R < \frac{kL}{4}$  and  $N_0 < N_\ell = \frac{L}{2} \left( 1 - \sqrt{1 - \frac{4R}{kL}} \right)$  extinction occurs at a time given by equation (6)

$$t_{ex} = \frac{1}{k\sqrt{1 - \frac{4R}{kL}}} \ln \left( \frac{N_\ell (N_u - N_0)}{N_u (N_\ell - N_0)} \right) = \frac{1}{k\sqrt{1 - \frac{4R}{kL}}} \ln \left[ \left( 1 - \frac{N_0}{N_u} \right) / \left( 1 - \frac{N_0}{N_\ell} \right) \right]. \quad (6)$$

An instructive exercise is to analyze the behavior of equation (6) in the limit of very large  $L$ . It should approach the expression for the extinction time derived from equation (3), namely  $t_{ex} = \frac{1}{k} \ln \left( \frac{R}{R - kN_0} \right)$ .

Now,  $N_\ell = \frac{R}{k} \left[ 1 + \frac{R}{kL} + O(L^{-2}) \right]$  and  $N_u = L \left[ 1 - \frac{R}{kL} + O(L^{-2}) \right]$ , so from the binomial expansion,

$$\frac{N_0}{N_u} = \frac{N_0}{L} + O(L^{-2}), \quad \frac{N_0}{N_\ell} = \frac{kN_0}{R} \left[ 1 - \frac{R}{kL} + O(L^{-2}) \right] = \frac{kN_0}{R} - \frac{N_0}{L} + O(L^{-2}) \quad \text{and} \quad \frac{1}{k\sqrt{1 - \frac{4R}{kL}}} = \frac{1}{k} \left( 1 + \frac{2R}{kL} + O(L^{-2}) \right).$$

The Maclaurin series for  $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \dots + \frac{(-1)^{n+1} x^n}{n}$ .

Thus, the logarithm factor in equation (6) has the following large  $L$  expansion.

$$\begin{aligned} \ln \left[ \left( 1 - \frac{N_0}{N_u} \right) / \left( 1 - \frac{N_0}{N_\ell} \right) \right] &= -\frac{N_0}{L} - \ln \left[ \left( 1 - \frac{kN_0}{R} \right) \left( 1 + N_0 / L \left( 1 - \frac{kN_0}{R} \right) \right) \right] + O(L^{-2}) \\ &= -\frac{N_0}{L} - \ln \left( \frac{R - kN_0}{R} \right) - \ln \left[ 1 + \frac{RN_0}{L(R - kN_0)} \right] + O(L^{-2}) \\ &= \ln \left( \frac{R}{R - kN_0} \right) - \frac{RN_0}{L(R - kN_0)} - \frac{N_0}{L} + O(L^{-2}) = \ln \left( \frac{R}{R - kN_0} \right) - \frac{N_0(2R - kN_0)}{L(R - kN_0)} + O(L^{-2}) \end{aligned}$$

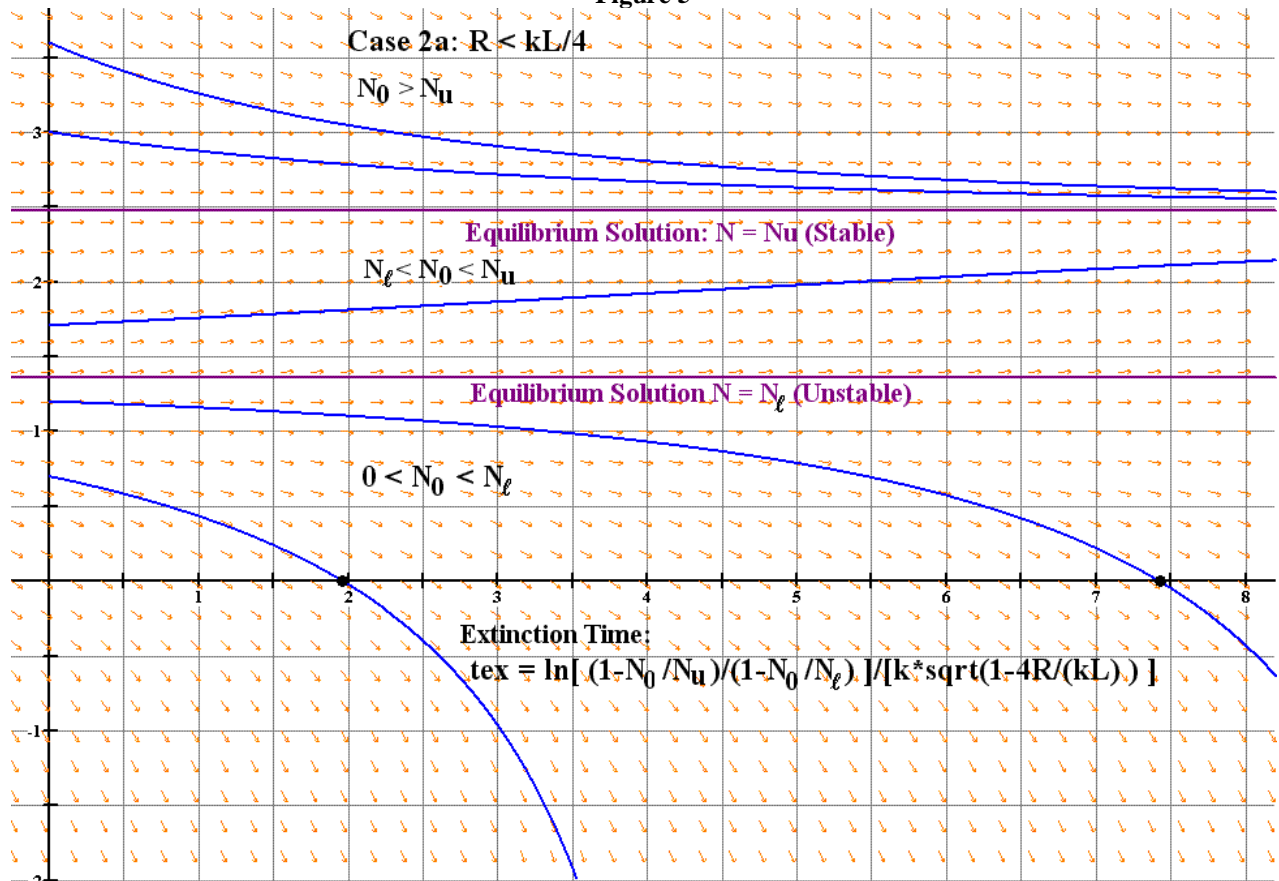
Therefore for large  $L$  the extinction time is given by the following formula.

$$\begin{aligned} t_{ex} &= \frac{1}{k} \left( 1 + \frac{2R}{kL} + O(L^{-2}) \right) \left[ \ln \left( \frac{R}{R - kN_0} \right) - \frac{N_0(2R - kN_0)}{L(R - kN_0)} + O(L^{-2}) \right] \\ &= \frac{1}{k} \left[ \ln \left( \frac{R}{R - kN_0} \right) + \frac{1}{L} \left( \frac{2R}{k} \ln \left( \frac{R}{R - kN_0} \right) - \frac{N_0(2R - kN_0)}{R - kN_0} \right) + O(L^{-2}) \right] \end{aligned}$$

This confirms the result from the solution of equation (1) in the limit that  $L$  goes to infinity.

Figure 5 summarizes the behavior of the solution of equation (1) for  $R < \frac{kL}{4}$ .

Figure 5



Case 2b:  $R = \frac{kL}{4}$ :

If  $R = \frac{kL}{4}$  there is only one equilibrium solution since then  $N_l = N_u = \frac{L}{2}$ . For  $N_0 \neq \frac{L}{2}$ ,  $\frac{dN}{dt} < 0$  and  $N(t)$  has no turning points since the solution can not pass through  $\frac{L}{2}$ . Therefore  $N(t)$  is a strictly decreasing function. If  $N_0 > \frac{L}{2}$ ,  $N(t) > \frac{L}{2}$  and so  $\lim_{t \rightarrow \infty} N(t)$  must exist which implies that  $\lim_{t \rightarrow \infty} N'(t) = 0$  and so  $\lim_{t \rightarrow \infty} N(t) = \frac{L}{2}$ . If  $N_0 < \frac{L}{2}$ ,  $N(t)$  must decrease without a lower limit and the population must become extinct. For  $R = \frac{kL}{4}$  equation (4) becomes  $\left(N - \frac{L}{2}\right)^{-2} dN = -\frac{k}{L} dt$ , which upon integrating becomes  $\left(N - \frac{L}{2}\right)^{-1} = \frac{k}{L} t + \left(N_0 - \frac{L}{2}\right)^{-1}$ . Solving for  $N(t)$  gives equation (7).

$$N(t) = \frac{L}{2} + \frac{1}{\frac{k}{L} t + \left(N_0 - \frac{L}{2}\right)^{-1}} = \frac{L}{2} + \frac{N_0 - \frac{L}{2}}{\frac{k}{L} \left(N_0 - \frac{L}{2}\right) t + 1}$$

$$N(t) = \frac{L}{2} + \frac{2N_0 - L}{k(2N_0/L - 1)t + 2} \quad (7)$$

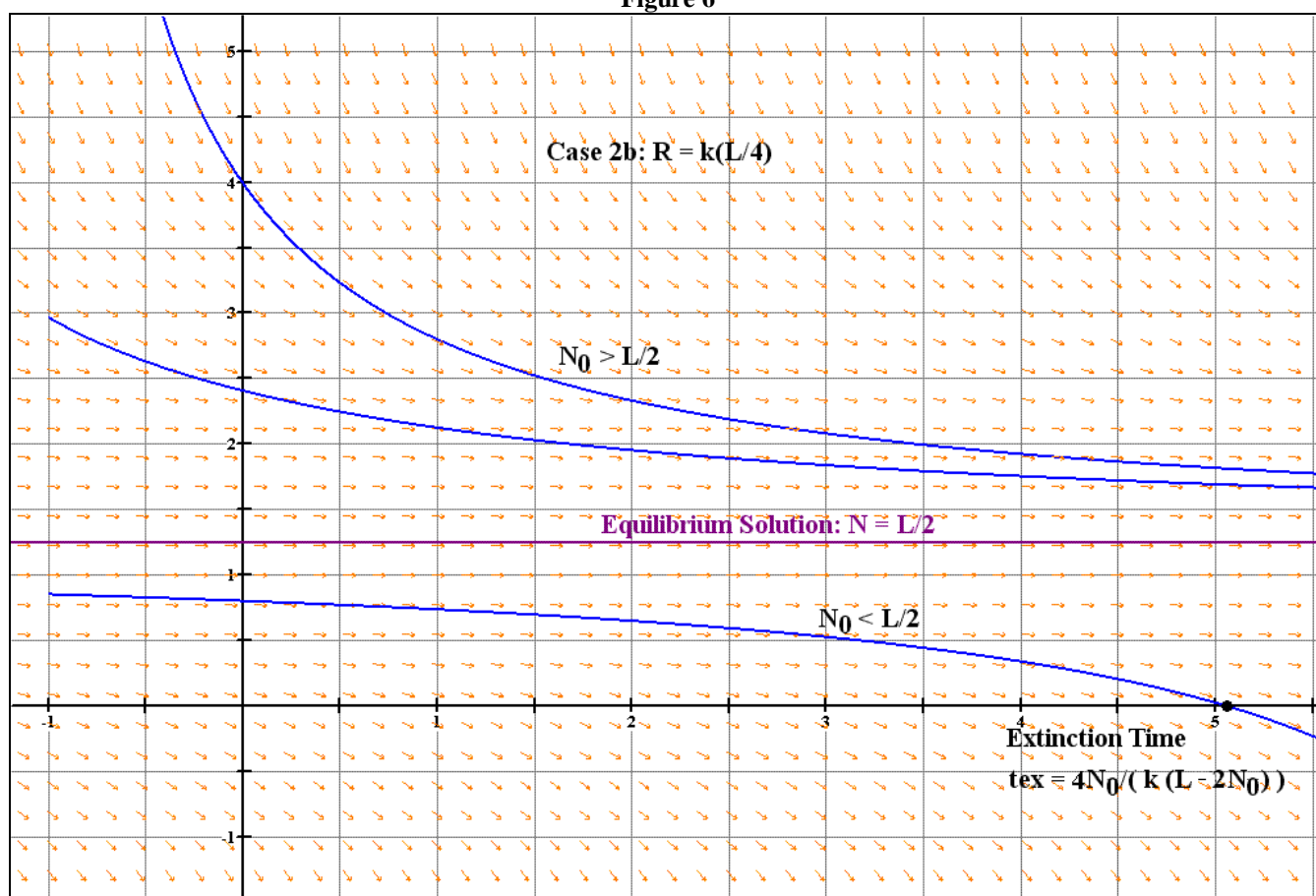
The  $\lim_{t \rightarrow \infty} N(t) = \frac{L}{2}$  for all values of  $N_0$ . If  $N_0 < \frac{L}{2}$  the denominator in equation (7) becomes negative for  $t > \frac{2L}{k(L-2N_0)}$ ,

so at  $t = t_c = \frac{2L}{k(L-2N_0)}$  the graph of  $N(t)$  has a vertical asymptote with  $\lim_{t \rightarrow t_c^-} N(t) = -\infty$ .  $N(t) = 0$  at  $t = t_{ex} < t_c$ . From equation (7)  $N(t_{ex}) = 0$  implies the following sequence of steps.

$$\begin{aligned} -\frac{L}{2} &= \frac{2N_0 - L}{k(2N_0/L - 1)t_{ex} + 2} \\ -Lk(2N_0/L - 1)t_{ex} - 2L &= 4N_0 - 2L \\ k(L - 2N_0)t_{ex} &= 4N_0 \\ t_{ex} &= \frac{4N_0}{k(L - 2N_0)} \quad (8) \end{aligned}$$

The case  $R = \frac{kL}{4}$  is illustrated in Figure 6.

Figure 6



If  $R < \frac{kL}{4}$ , then equation (5) is valid. Consider the  $\lim_{R \rightarrow \frac{kL}{4}^-} N(t) = \lim_{R \rightarrow \frac{kL}{4}^-} N_u + \frac{(N_u - N_\ell)(N_0 - N_u)}{(N_0 - N_\ell) \exp\left(k\sqrt{1 - \frac{4R}{kL}} t\right) + N_u - N_0}$ .

Let  $x = \sqrt{1 - \frac{4R}{kL}}$  as  $R \rightarrow \frac{kL}{4}^-$ ,  $x \rightarrow 0^+$ .  $N_\ell = \frac{L}{2}(1-x)$  and  $N_u = \frac{L}{2}(1+x)$  so that the limit can be evaluated as follows.

$$\begin{aligned}
\lim_{R \rightarrow \frac{kL^-}{4}} N(t) &= \lim_{x \rightarrow 0^+} \frac{L}{2}(1+x) + \frac{Lx \left( N_0 - \frac{L}{2}(1+x) \right)}{\left( N_0 - \frac{L}{2}(1-x) \right) \exp(kt) + \frac{L}{2}(1+x) - N_0} \\
&= \frac{L}{2} + \lim_{x \rightarrow 0^+} \frac{Lx \left( N_0 - \frac{L}{2}(1+x) \right)}{\left( N_0 - \frac{L}{2} \right) [\exp(kt) - 1] + x \frac{L}{2} [1 + \exp(kt)]} \\
&= \frac{L}{2} + \lim_{x \rightarrow 0^+} \frac{Lx \left( N_0 - \frac{L}{2}(1+x) \right)}{\left( N_0 - \frac{L}{2} \right) [kxt + O(x^2)] + x \frac{L}{2} [1 + 1 + kxt + O(x^2)]} \\
&= \frac{L}{2} + \lim_{x \rightarrow 0^+} \frac{L \left( N_0 - \frac{L}{2}(1+x) \right)}{\left( N_0 - \frac{L}{2} \right) [kt + O(x)] + \frac{L}{2} [2 + kxt + O(x^2)]} \\
&= \frac{L}{2} + \frac{L \left( N_0 - \frac{L}{2} \right)}{\left( N_0 - \frac{L}{2} \right) kt + L} = \frac{L}{2} + \frac{L(2N_0 - L)}{(2N_0 - L)kt + 2L} \\
&= \frac{L}{2} + \frac{(2N_0 - L)}{(2N_0/L - 1)kt + 2}
\end{aligned}$$

This last line is identical to equation (7), so the required continuity of the solutions with respect to the parameter  $R$  has been confirmed.

Similarly, using the Maclaurin series for  $\ln(1+x)$  the limit as  $R \rightarrow \frac{kL^-}{4}$  of the extinction time in equation (6) can be expressed as follows:

$$\begin{aligned}
\lim_{R \rightarrow \frac{kL^-}{4}} t_{ex} &= \lim_{x \rightarrow 0^+} \frac{1}{kx} \ln \left( \frac{\frac{L}{2}(1-x) \left( \frac{L}{2}(1+x) - N_0 \right)}{\frac{L}{2}(1+x) \left( \frac{L}{2}(1-x) - N_0 \right)} \right) = \lim_{x \rightarrow 0^+} \frac{1}{kx} \ln \left( \frac{(1-x) \left( 1+x - \frac{2N_0}{L} \right)}{(1+x) \left( 1-x - \frac{2N_0}{L} \right)} \right) \\
&= \frac{1}{k} \lim_{x \rightarrow 0^+} \frac{\ln(1-x) + \ln \left[ \left( 1 - \frac{2N_0}{L} \right) \left( 1 + \frac{x}{1 - \frac{2N_0}{L}} \right) \right] - \ln(1+x) - \ln \left[ \left( 1 - \frac{2N_0}{L} \right) \left( 1 - \frac{x}{1 - \frac{2N_0}{L}} \right) \right]}{x} \\
&= \frac{1}{k} \lim_{x \rightarrow 0^+} \frac{\ln(1-x) + \ln \left( 1 - \frac{2N_0}{L} \right) + \ln \left( 1 + \frac{x}{1 - \frac{2N_0}{L}} \right) - \ln(1+x) - \ln \left( 1 - \frac{2N_0}{L} \right) - \ln \left( 1 - \frac{x}{1 - \frac{2N_0}{L}} \right)}{x}
\end{aligned}$$

$$\lim_{R \rightarrow \frac{kL}{4}} t_{ex} = \frac{1}{k} \lim_{x \rightarrow 0^+} \frac{-x + \frac{Lx}{L-2N_0} - x + \frac{Lx}{L-2N_0} + O(x^2)}{x} = \frac{2}{k} \left( \frac{L}{L-2N_0} - 1 \right) = \frac{2}{k} \left( \frac{L-L+2N_0}{L-2N_0} \right)$$

$$= \frac{4N_0}{k(L-2N_0)}$$

This limit, as expected, is identical to equation (8).

**Case 2c:**  $R > \frac{kL}{4}$ :

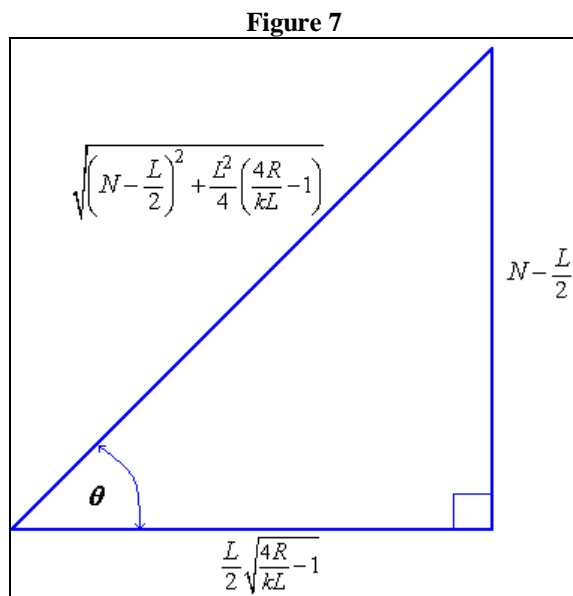
If  $R > \frac{kL}{4}$  there are no real equilibrium solutions. Equation (4) becomes

$$\frac{dN}{\left(N - \frac{L}{2}\right)^2 + \frac{L^2}{4} \left(\frac{4R}{kL} - 1\right)} = -\frac{k}{L} dt$$

So  $\frac{dN}{dt} < 0$  and there are no turning points and no horizontal asymptotes. Thus,  $N(t)$  must decrease without a lower limit

and the population must become extinct. This is true for every initial population  $N_0$ .  $\left(N(t) - \frac{L}{2}\right)^2 + \frac{L^2}{4} \left(\frac{4R}{kL} - 1\right) > 0$ ; this

motivates the trigonometric substitution  $N = \frac{L}{2} + \tan(\theta) \frac{L}{2} \sqrt{\frac{4R}{kL} - 1}$ .



$$dN = \frac{L}{2} \sqrt{\frac{4R}{kL} - 1} \sec^2(\theta) d\theta \text{ and } \left(\frac{L}{2} - N(t)\right)^2 + \frac{L^2}{4} \left(\frac{4R}{kL} - 1\right) = \frac{L^2}{4} \left(\frac{4R}{kL} - 1\right) \sec^2(\theta). \text{ Equation (4) then becomes}$$

$$\frac{\frac{d\theta}{\frac{L}{2}\sqrt{\frac{4R}{kL}-1}}}{-\frac{k}{L}dt} \text{ which integrates as } \theta = -\frac{kt}{2}\sqrt{\frac{4R}{kL}-1} + \theta_0 \text{ or } \tan^{-1}\left(\frac{N-\frac{L}{2}}{\frac{L}{2}\sqrt{\frac{4R}{kL}-1}}\right) = -\frac{kt}{2}\sqrt{\frac{4R}{kL}-1} + \tan^{-1}\left(\frac{N_0-\frac{L}{2}}{\frac{L}{2}\sqrt{\frac{4R}{kL}-1}}\right).$$

Solving for  $N(t)$

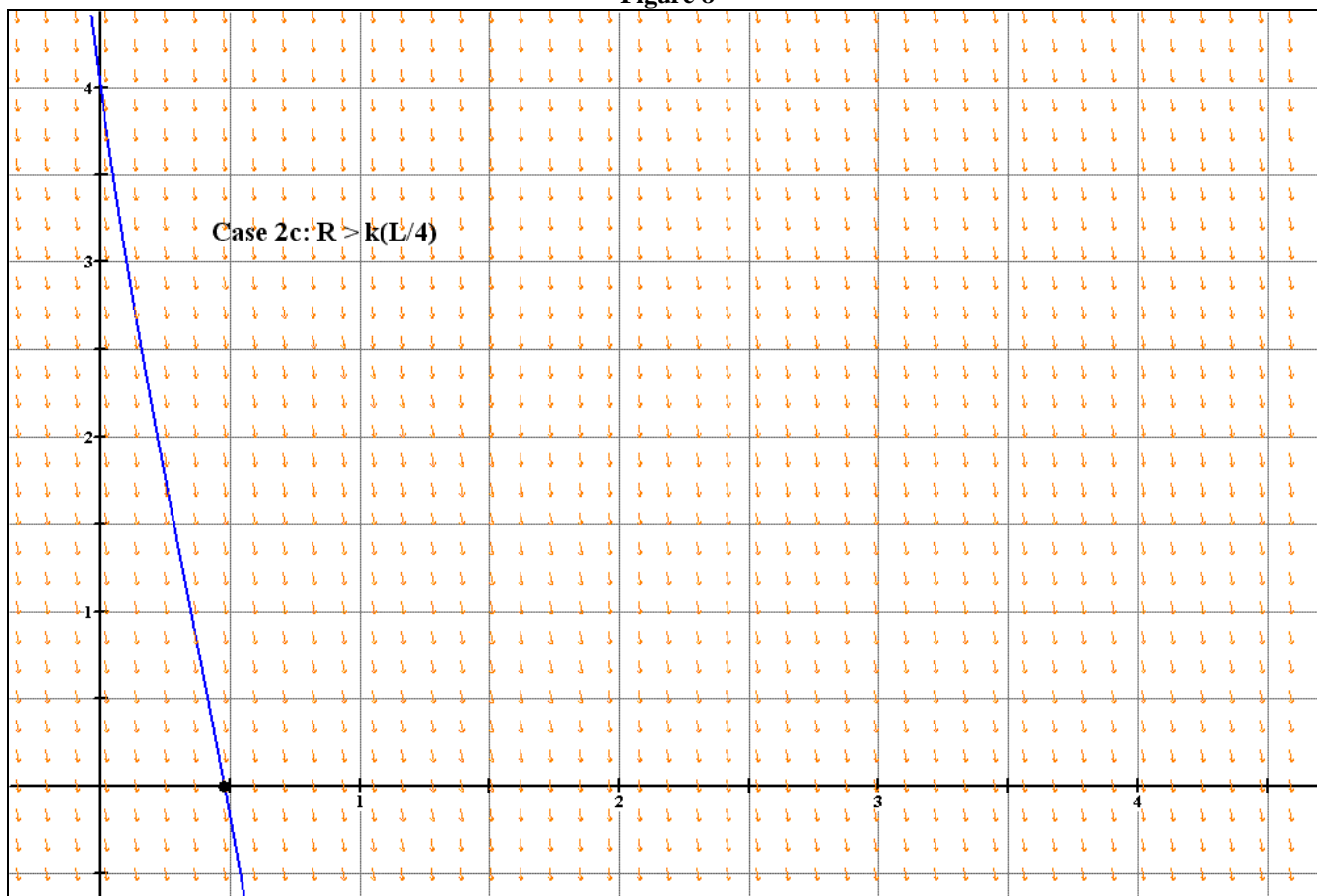
$$\begin{aligned} \tan^{-1}\left(\frac{N-\frac{L}{2}}{\frac{L}{2}\sqrt{\frac{4R}{kL}-1}}\right) &= -\frac{kt}{2}\sqrt{\frac{4R}{kL}-1} + \tan^{-1}\left(\frac{2N_0-L}{L\sqrt{\frac{4R}{kL}-1}}\right) \\ N(t) &= \frac{L}{2} + \frac{L}{2}\sqrt{\frac{4R}{kL}-1} \tan\left[\tan^{-1}\left(\frac{2N_0-L}{L\sqrt{\frac{4R}{kL}-1}}\right) - \frac{kt}{2}\sqrt{\frac{4R}{kL}-1}\right] \\ N(t) &= \frac{L}{2} \left\{ 1 + \sqrt{\frac{4R}{kL}-1} \tan\left[\tan^{-1}\left(\frac{2N_0-L}{L\sqrt{\frac{4R}{kL}-1}}\right) - \frac{kt}{2}\sqrt{\frac{4R}{kL}-1}\right] \right\} \end{aligned} \quad (9)$$

Extinction occurs for all positive initial populations. The extinction time is obtained as follows.

$$\begin{aligned} \tan\left[\tan^{-1}\left(\frac{2N_0-L}{L\sqrt{\frac{4R}{kL}-1}}\right) - \frac{k t_{ex}}{2}\sqrt{\frac{4R}{kL}-1}\right] &= -\frac{1}{\sqrt{\frac{4R}{kL}-1}} \\ \frac{k t_{ex}}{2}\sqrt{\frac{4R}{kL}-1} - \tan^{-1}\left(\frac{2N_0-L}{L\sqrt{\frac{4R}{kL}-1}}\right) &= \tan^{-1}\left(\frac{1}{\sqrt{\frac{4R}{kL}-1}}\right) \\ \frac{k t_{ex}}{2}\sqrt{\frac{4R}{kL}-1} &= \tan^{-1}\left(\frac{1}{\sqrt{\frac{4R}{kL}-1}}\right) + \tan^{-1}\left(\frac{2N_0-L}{L\sqrt{\frac{4R}{kL}-1}}\right) \\ t_{ex} &= \frac{2}{k\sqrt{\frac{4R}{kL}-1}} \left[ \tan^{-1}\left(\frac{1}{\sqrt{\frac{4R}{kL}-1}}\right) + \tan^{-1}\left(\frac{2N_0-L}{L\sqrt{\frac{4R}{kL}-1}}\right) \right] \end{aligned} \quad (10)$$

The case  $R > \frac{kL}{4}$  is illustrated in Figure 8.

Figure 8



From equation (9) consider the following limit.

$$\lim_{R \rightarrow \frac{kL^+}{4}} N(t) = \lim_{R \rightarrow \frac{kL^+}{4}} \frac{L}{2} \left\{ 1 + \sqrt{\frac{4R}{kL} - 1} \tan \left[ \tan^{-1} \left( \frac{2N_0 - L}{L\sqrt{\frac{4R}{kL} - 1}} \right) - \frac{kt}{2} \sqrt{\frac{4R}{kL} - 1} \right] \right\}.$$

Let  $B = \sqrt{\frac{4R}{kL} - 1}$  as  $R \rightarrow \frac{kL^+}{4}$ ,  $B \rightarrow 0^+$ . Using the addition formula for the tangent function and the fundamental result that

$$\lim_{x \rightarrow 0} \frac{\tan(\alpha x)}{x} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx} [\tan(\alpha x)]}{\frac{d}{dx}(x)} = \lim_{x \rightarrow 0} \frac{\sec^2(\alpha x) \alpha}{1} = \alpha, \text{ the } \lim_{R \rightarrow \frac{kL^+}{4}} N(t) \text{ is calculated as follows.}$$

$$\begin{aligned} \lim_{R \rightarrow \frac{kL^+}{4}} N(t) &= \lim_{B \rightarrow 0^+} \frac{L}{2} \left\{ 1 + B \tan \left[ \tan^{-1} \left( \frac{2N_0 - L}{LB} \right) - \frac{kt}{2} B \right] \right\} \\ &= \frac{L}{2} + \frac{L}{2} \lim_{B \rightarrow 0^+} B \tan \left[ \tan^{-1} \left( \frac{2N_0 - L}{LB} \right) - \frac{kt}{2} B \right] \end{aligned}$$

$$\begin{aligned}
\lim_{R \rightarrow \frac{kL}{4}^+} N(t) &= \frac{L}{2} + \frac{L}{2} \lim_{B \rightarrow 0^+} B \frac{\tan \left[ \tan^{-1} \left( \frac{2N_0 - L}{LB} \right) \right] + \tan \left[ -\frac{kt}{2} B \right]}{1 - \tan \left[ \tan^{-1} \left( \frac{2N_0 - L}{LB} \right) \right] \tan \left[ -\frac{kt}{2} B \right]} \\
&= \frac{L}{2} + \frac{L}{2} \lim_{B \rightarrow 0^+} \frac{\frac{B(2N_0 - L)}{LB} + B \tan \left[ -\frac{kt}{2} B \right]}{1 - \frac{2N_0 - L}{L} \tan \left[ -\frac{kt}{2} B \right]} / B \\
&= \frac{L}{2} + \left( \frac{L}{2} \right) \frac{\frac{2N_0 - L}{L}}{1 - \frac{2N_0 - L}{L} \left( -\frac{kt}{2} \right)} = \frac{L}{2} + \frac{2N_0 - L}{2 + k \left( \frac{2N_0 - L}{L} \right) t} \\
&= \frac{L}{2} + \frac{2N_0 - L}{k(2N_0/L - 1)t + 2}
\end{aligned}$$

This last line is identical to equation (7). So once again the required continuity of the solutions with respect to the parameter  $R$  has been confirmed.

When  $R = \frac{kL}{4}$  and  $N_0 < \frac{L}{2}$ , the extinction time was given by equation (8),  $t_{ex} = \frac{4N_0}{k(L - 2N_0)}$ . This should match the limit

of equation (10) as  $R \rightarrow \frac{kL}{4}^+$ .

$$\lim_{R \rightarrow \frac{kL}{4}^+} t_{ex} = \lim_{B \rightarrow 0^+} \frac{2}{kB} \left[ \tan^{-1} \left( \frac{1}{B} \right) + \tan^{-1} \left( \frac{2N_0/L - 1}{B} \right) \right] = \lim_{B \rightarrow 0^+} \frac{2}{kB} \left[ \tan^{-1} \left( \frac{1}{B} \right) - \tan^{-1} \left( \frac{1 - 2N_0/L}{B} \right) \right].$$

Now for positive  $\alpha$  consider the following function defined for  $x \geq 0$ .

$$g(x) = \begin{cases} \tan^{-1} \left( \frac{\alpha}{x} \right) & \text{if } x > 0 \\ \frac{\pi}{2} & \text{if } x = 0 \end{cases}$$

$$g'(x) = \frac{1}{1 + \alpha^2 x^{-2}} \left( -\alpha x^{-2} \right) = \frac{-\alpha}{x^2 + \alpha^2}$$

$$\lim_{x \rightarrow 0^+} g(x) = g(0) = \frac{\pi}{2}$$

$$g'(0) = -\alpha^{-1}$$

This function has the expansion  $g(x) = \frac{\pi}{2} - \frac{x}{\alpha} + O(x^2)$  for small  $x$ . Thus, for small positive  $B$  the following expansions result.

$$\tan^{-1} \left( \frac{1}{B} \right) = \frac{\pi}{2} - B + O(B^2)$$

$$\tan^{-1} \left( \frac{1 - 2N_0/L}{B} \right) = \frac{\pi}{2} - \frac{B}{1 - 2N_0/L} + O(B^2)$$

$$\begin{aligned} & \frac{2}{kB} \left[ \tan^{-1} \left( \frac{1}{B} \right) - \tan^{-1} \left( \frac{1-2N_0/L}{B} \right) \right] = \frac{2}{kB} \left[ \frac{\pi}{2} - B - \frac{\pi}{2} + \frac{B}{1-2N_0/L} + O(B^2) \right] \\ & = \frac{2}{k} \left[ \frac{1}{1-2N_0/L} - 1 + O(B) \right] = \frac{2}{k} \left[ \frac{1-1+2N_0/L}{1-2N_0/L} + O(B) \right] = \frac{2}{k} \left[ \frac{2N_0}{L-2N_0} + O(B) \right] \end{aligned}$$

So the limit of the extinction time becomes

$$\lim_{R \rightarrow \frac{kL}{4}^+} t_{ex} = \lim_{B \rightarrow 0^+} \frac{2}{kB} \left[ \tan^{-1} \left( \frac{1}{B} \right) - \tan^{-1} \left( \frac{1-2N_0/L}{B} \right) \right] = \frac{2}{k} \left( \frac{2N_0}{L-2N_0} \right) = \frac{4N_0}{k(L-2N_0)}.$$

As was expected from the continuity of the solution of equation (1) with respect to the parameter  $R$  this limit is identical to equation (8).

The following table summarizes the major results of this analysis.

<b>Summary of the Properties of the Logistic Equation with a Harvesting Term <math>R</math> :</b> $\frac{dN}{dt} = kN \left( 1 - \frac{N}{L} \right) - R$ ; $N(0) = N_0$			
Case	Solution for $N(t)$	Condition for Extinction	Extinction Time
$L \rightarrow \infty$	$\frac{R}{k} + \left( N_0 - \frac{R}{k} \right) e^{kt}$	$N_0 < \frac{R}{k}$	$\frac{1}{k} \ln \left( \frac{R}{R - kN_0} \right)$
$R < \frac{kL}{4}$	$N_u + \frac{(N_u - N_\ell)(N_0 - N_u)}{(N_0 - N_\ell) \exp \left( k \sqrt{1 - \frac{4R}{kL}} t \right) + N_u - N_0}$ <p>where</p> $N_\ell = \frac{L}{2} \left( 1 - \sqrt{1 - \frac{4R}{kL}} \right)$ $N_u = \frac{L}{2} \left( 1 + \sqrt{1 - \frac{4R}{kL}} \right)$	$N_0 < N_\ell$	$\frac{\ln \left[ \left( 1 - \frac{N_0}{N_u} \right) / \left( 1 - \frac{N_0}{N_\ell} \right) \right]}{k \sqrt{1 - \frac{4R}{kL}}}$
$R = \frac{kL}{4}$	$\frac{L}{2} + \frac{2N_0 - L}{k(2N_0/L - 1)t + 2}$	$N_0 < \frac{L}{2}$	$\frac{4N_0}{k(L - 2N_0)}$
$R > \frac{kL}{4}$	$\frac{L}{2} \left\{ 1 + B \tan \left( \tan^{-1} \left( \frac{2N_0 - L}{LB} \right) - \frac{kBt}{2} \right) \right\}$ <p>where</p> $B = \sqrt{\frac{4R}{kL} - 1}$	Any value of $N_0$	$\frac{2}{kB} \left[ \tan^{-1} \left( \frac{1}{B} \right) + \tan^{-1} \left( \frac{2N_0 - L}{LB} \right) \right]$