Bonus Project 1 **A Deranged Experiment** /15
Name ________________________ **Due 3/12/2009**

Consider *n* letters with *n* different address headings and *n* letters each addressed to one of these addresses. Suppose before the letters are stuffed into the envelopes, they are dropped on the floor and then carelessly (randomly) assigned to the *n* envelopes. Another statement of essentially the same problem is as follows: *n* men check in their hats at a club. Due to employee shortages at the club, as the men leave they are randomly given a hat from the collection of *n* hats. A more dramatic but equivalent scenario might have *n* patients' medical records and *n* lab results. Because of some mix up, the tests results are randomly assigned to the medical records. An assignment of the *n* objects (letters, test results, etc.) to *n* receptacles (envelopes, lab reports, etc.) such that **none** of them is in the proper position is called a derangement of the *n* objects. A more abstract algebraic definition of a derangement of *n* objects is a permutation of order *n* which has no fixed points. Over two hundred fifty years ago Leonhard Euler derived a formula for the number of different derangements of *n* objects in the context of analyzing the probability of winning a card game called Rencontre.

The Experiment:
Write the numbers 1 through 16 on 16 identical (as close as you can make them) slips of paper. Be sure to distinguish a 6 from a 9 . Put these slips into an envelope, stir them up and randomly draw out each slip, one at a time. As each slip is drawn, place it on a marked four by four grid such as the one shown below. The first slip goes in the cell marked 1, the second in the cell marked 2, and so on. Record how many of the slip numbers match the grid number of the slip. The number of "correct" matches, *x*, is the relevant random variable.

1	**2**	**3**	**4**
5	**6**	**7**	**8**
9	**10**	**11**	**12**
13	**14**	**15**	**16**

Repeat the experiment 100 times and record your results in Table 1.

The Analysis
In order to determine the number of derangements and the probability distribution of *x*, consider the following line of thought. Let $C(x,n)$ be the number of distinct ways that there are *x* correct matches out of *n*. For convenience adopt the definition that $C(0,0) \equiv 1$. The following values should be "obvious"; fill them in.

A. $C(n,n) =$ ___________ $C(0,1) =$ ___________ $C(n-1,n) =$ ___________

Table 1

Event	**Observed *f***	**Empirical Probability**	**Theoretical Probability**
$x = 0$			
$x = 1$			
$x = 2$			
$x > 2$			
	μ_x		
	σ_x		

To determine a recursion for $C(0, n)$, let $E(n)$ designate the number of distinct one-to-one functions, f_i, from the set $\{e_1, e_2, e_3, \ldots e_{n-1}, q\}$ onto the set $\{e_1, e_2, e_3, \ldots e_{n-1}, e_n\}$ where $q \neq e_n$ and $f_i(s) \neq s$ for s in $\{e_1, e_2, e_3, \ldots e_{n-1}\}$. Since there is only one function from $\{q\}$ onto $\{e_1\}$, $E(1) = 1$. Similarly, there is only one function from $\{e_1, q\}$ onto $\{e_1, e_2\}$, for which $f(s) \neq s$, namely, $f(e_1) = e_2$, and $f(q) = e_1$. So, $E(2) = 1$. As is shown below there are exactly three functions from $\{e_1, e_2, q\}$ onto $\{e_1, e_2, e_3\}$ with $f_i(s) \neq s$. So, $E(3) = 3$.

$f_1(q) = e_1$	$f_2(q) = e_2$	$f_3(q) = e_3$
$f_1(e_1) = e_2$	$f_2(e_1) = e_3$	$f_3(e_1) = e_2$
$f_1(e_2) = e_3$	$f_2(e_2) = e_1$	$f_3(e_2) = e_1$

B. Develop a recursion that expresses $E(n)$ in terms of $C(0, n-1)$ and $E(n-1)$.

C. Develop a recursion that expresses $C(0, n)$ in terms of $E(n-1)$.

D. From the recursions in B and C., develop a recursion that expresses $C(0, n)$ in terms of $C(0, n-1)$ and $C(0, n-2)$.

E. Use your recursion to fill in the following.

$C(0,0) =$____1____ $C(0,1) =$____0____ $C(0,2) =$_________ $C(0,3) =$_________

$C(0,4) =$_________ $C(0,5) =$_________ $C(0,6) =$_________ $C(0,7) =$_________

F. Using the results from E., guess a formula for $C\left(0,n\right) - nC\left(0,n-1\right)$. Verify your guess using mathematical induction.

G. $C(0,n)$ can be expressed as the sum, $C(0,n) = n!\sum_{j=0}^{n} a_j$. Using the result from F., determine the formula for a_j .

H. Using the result from G., evaluate $\lim_{n\to\infty} \frac{C(0,n)}{n!} = .$

I. Using the result from G. and the properties of binomial coefficients, give an explicit formula for $P\left(x,n\right)$, the probability of *x* correct matches out of *n*.

J. Using the probability derived in I., calculate the moments $\langle x \rangle$ and $\langle x^2 \rangle$ and from them the mean, μ_x, and standard deviation, σ_x, of the theoretical probability distribution.

K. In Table 1, how do the empirical and theoretical event probabilities compare? How do the empirical and theoretical means compare? How do the empirical and theoretical standard deviations compare?

The Solution:

Step 1

To determine the number of distinct one-to-one functions, f_i , from the set $\{e_1, e_2, e_3, \dots e_{n-1}, q\}$ onto the set $\{e_1, e_2, e_3, \dots e_{n-1}, e_n\}$ where $q \neq e_n$ and $f_i(s) \neq s$ for *s* in $\{e_1, e_2, e_3, \dots e_{n-1}\}$, consider first the number of such functions which map *q* onto e_n . From the definition of *C*(*x*, *n*) this can happen *C*(0, *n*−1) distinct ways. If $f(q) \neq e_n$, then $f(q) = e_\ell$ with $e_\ell \neq e_n$ for *n*−1 different choices for e_ℓ . But each such *f* must also map $\{e_1, e_2, e_3, \dots e_{n-1}\}$ onto $\{e_1, e_2, e_3, \dots e_{n-1}, e_n\} - \{e_\ell\}$. Since $e_\ell \in \{e_1, e_2, e_3, \dots, e_{n-1}\}$, e_ℓ is the "new" *q* and so there are *E*(*n*−1) ways this can happen. Thus, there are (*n*−1)*E*(*n*−1) distinct functions which map $\{e_1, e_2, e_3, \dots e_{n-1}, q\}$ onto $\{e_1, e_2, e_3, \dots e_{n-1}, e_n\}$ with no fixed points and $f(q) \neq e_n$. Thus,

$$E(n) = C(0, n-1) + (n-1)E(n-1) \qquad \textbf{(S1)}$$

Step 2

Consider a bijection *f* from $\{e_1, e_2, e_3, \dots e_{n-1}, e_n\}$ to $\{e_1, e_2, e_3, \dots e_{n-1}, e_n\}$. If *f* has no fixed points, $f(e_1) = e_\ell \neq e_1$ for *n*−1 choices of e_ℓ . Now *f* must also map $\{e_2, e_3, \dots e_n\}$ one-to-one onto $\{e_1, e_2, e_3, \dots e_{n-1}, e_n\} - \{e_\ell\}$. Since $e_\ell \in \{e_2, e_3, \dots e_n\}$, there are *E*(*n*−1) different one-to-one maps with no fixed points from $\{e_2, e_3, \dots e_n\}$ onto $\{e_1, e_2, e_3, \dots e_{n-1}, e_n\} - \{e_\ell\}$. Thus, the total number of bijections from $\{e_1, e_2, e_3, \dots e_{n-1}, e_n\}$ to $\{e_1, e_2, e_3, \dots e_{n-1}, e_n\}$ with no fixed points is (*n*−1)*E*(*n*−1).

$$C(0, n) = (n-1)E(n-1) \qquad \textbf{(S2)}$$

Step 3

Plugging Equation **(S2)** into Equation **(S1)** yields

$$E(n) = C(0, n-1) + C(0, n) .$$

Plugging this result back into Equation **(S2)** in turn gives the following two term recursion for *C*(0, *n*)

$$C(0, n) = (n-1)\left[C(0, n-1) + C(0, n-2)\right] \qquad \textbf{(S3)}$$

Step 4

Equation **(S3)** with the initial values $C(0,0) = 1$ and $C(0,1) = 0$ completely determines all of the values of *C*(0, *n*).Using the recursion we obtain the following:

$$C(0,2) = 1(0+1) = 1 = 2(0)+1$$
$$C(0,3) = 2(1+0) = 2 = 3(1)-1$$
$$C(0,4) = 3(2+1) = 9 = 4(2)+1$$
$$C(0,5) = 4(9+2) = 44 = 5(9)-1$$
$$C(0,6) = 5(44+9) = 265 = 6(44)+1$$
$$C(0,7) = 6(265+44) = 1854 = 7(265)-1$$
$$C(0,8) = 7(1854+265) = 14{,}833 = 8(1854)+1$$

Step 5
Verify using mathematical induction that

$$C(0,n)=(n)C(0,n-1)+(-1)^n . \qquad \textbf{(S4)}$$

Base case: $(1)C(0,1-1)+(-1)^1=1+(-1)=0=C(0,1)$
Induction case:
If $C(0,n-1)=(n-1)C(0,n-2)+(-1)^{n-1}$ then from Equation **(S3)**

$$C(0,n)=(n-1)\left[C(0,n-1)+C(0,n-2)\right]=(n-1)C(0,n-1)+(n-1)C(0,n-2)$$

$$=(n-1)C(0,n-1)-(-1)^{n-1}+\left[(n-1)C(0,n-2)+(-1)^{n-1}\right]$$

$$=(n-1)C(0,n-1)+(-1)^n+C(0,n-1)=(n)C(0,n-1)+(-1)^n$$

Step 6
Suppose $C(0,n)=\sum_{j=0}^{n}a(j,n)$, then from Equation **(S4)**, $\sum_{j=0}^{n}a(j,n)=\sum_{j=0}^{n-1}na(j,n-1)+(-1)^n$.
This has as a solution

$$a(n,n)=(-1)^n$$

$$a(n-1,n)=na(n-1,n-1)=n(-1)^{n-1}$$

$$a(n-2,n)=na(n-2,n-1)=n(n-1)(-1)^{n-2}$$

$$a(n-3,n)=na(n-3,n-1)=n(n-1)(n-2)(-1)^{n-3}$$

$\vdots$

$$a(n-j,n)=na(n-j,n-1)=n(n-1)(n-2)(n-3)\cdots(n-j+1)(-1)^{n-j}=\frac{n!}{(n-j)!}(-1)^{n-j}$$

Or, more simply $a(j,n)=\frac{n!}{(j)!}(-1)^j$, so

$$C(0,n)=n!\sum_{j=0}^{n}\frac{(-1)^j}{j!} \qquad \textbf{(S5)}$$

Note 1: $\lim_{n\to\infty}\frac{C(0,n)}{n!}=\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}=e^{-1}=\frac{1}{e}$.

Note 2: $C(0,n)$ is the number of permutation matrices of order *n* with zero trace. This is given asymptotically by $\frac{n!}{e}$.

Step 7
$C(x,n)$ is the number of distinct ways that there are *x* correct matches out of *n*. A bijection from from $\{e_1,e_2,e_3,\ldots e_{n-1},e_n\}$ to $\{e_1,e_2,e_3,\ldots e_{n-1},e_n\}$ that maps *x* elements to themselves must also map the remaining *n* – *x* elements to the same *n* – *x* elements with no fixed points. This can

happen $C(0, n-x)$ ways. The number of distinct choices for the *x* elements out of *n* which are mapped to themselves is the binomial coefficient $\binom{n}{x}$. Thus,

$$C(x,n) = \binom{n}{x} C(0, n-x) = \frac{n!}{x!} \sum_{j=0}^{n-x} \frac{(-1)^j}{j!} \quad \textbf{(S6)}$$

Note 3: $C(x,n)$ is the number of permutation matrices of order *n* with trace equal to *x*.

Step 8

Since there are *n*! bijections from *n* distinct objects onto themselves, assuming a random selection, the probability of *x* correct matches out of *n* is given by

$$P(x,n) = \frac{C(x,n)}{n!} = \frac{1}{x!} \sum_{j=0}^{n-x} \frac{(-1)^j}{j!} . \qquad \textbf{(S7)}$$

Note 4: For large *n* (and the factorial in the denominator means this is not too large!), the probability of zero matches and the probability of exactly one match are both given approximately by $\frac{1}{e}$. By the Leibniz's alternating series theorem, $P(0,n) = \sum_{j=0}^{n} \frac{(-1)^j}{j!}$ differs from $\frac{1}{e}$ by no more than $\frac{1}{(n+1)!}$ and $P(1,n) = \sum_{j=0}^{n-1} \frac{(-1)^j}{j!}$ differs from $\frac{1}{e}$ by no more than $\frac{1}{n!}$.

Step 9

To calculate the mean and standard deviation of the probability distribution of the number of correct matches out of *n*, consider the following generating function.

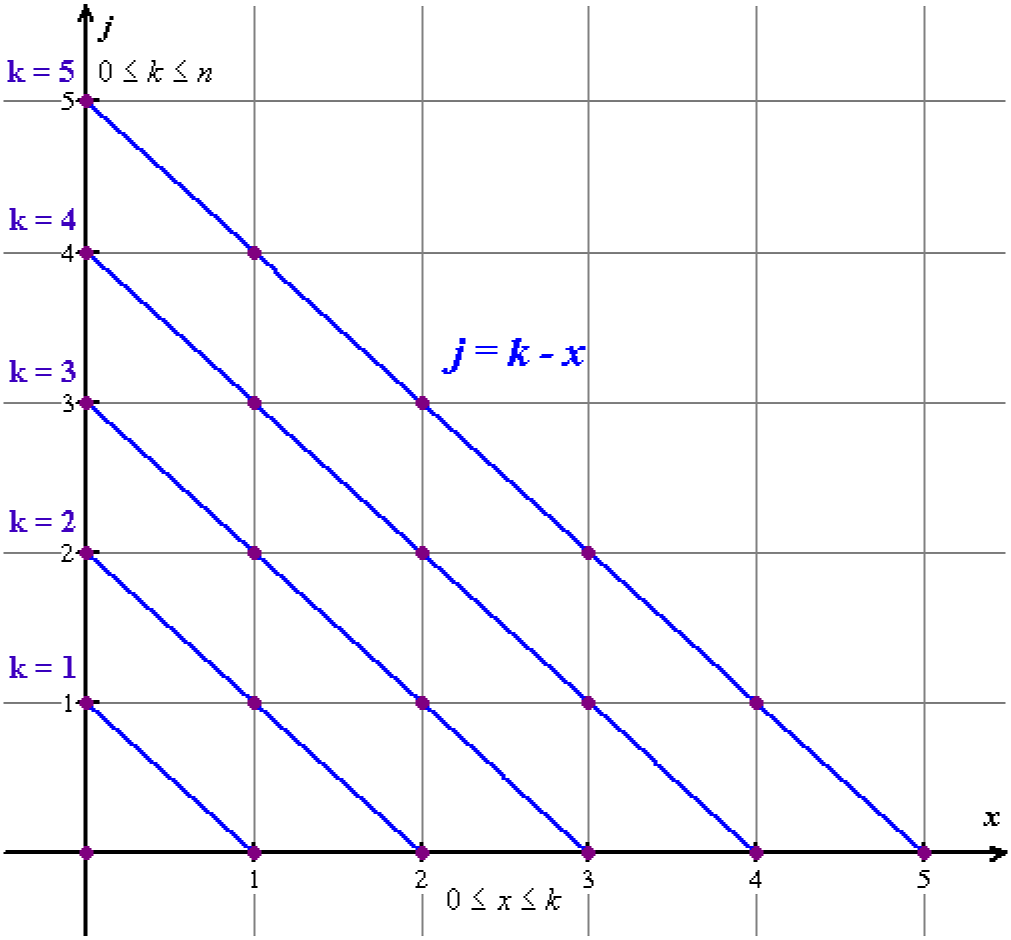
$$G(z,n) = \sum_{x=0}^{n} \frac{z^x}{x!} \sum_{j=0}^{n-x} \frac{(-1)^j}{j!} . \qquad \textbf{(S8)}$$

Rearranging the terms as shown in the figure below with the *j* intercept, *k*, running from $0 \le k \le n$, *x* running from $0 \le x \le k$, with $j = k - x$ gives

$$G(z,n) = \sum_{k=0}^{n} \sum_{x=0}^{k} \frac{z^x}{x!} \frac{(-1)^{k-x}}{(k-x)!} = \sum_{k=0}^{n} \frac{1}{k!} \sum_{x=0}^{k} \frac{k!}{x!(k-x)!} z^x (-1)^{k-x} = \sum_{k=0}^{n} \frac{1}{k!} \sum_{x=0}^{k} \binom{k}{x} z^x (-1)^{k-x}$$

From the Binomial Theorem this simplifies to

$$G(z,n) = 1 + \sum_{k=1}^{n} \frac{(z-1)^k}{k!} \qquad \textbf{(S9)}$$



Putting these results together,

$$G(z,n)=\sum_{x=0}^{n}\frac{z^x}{x!}\sum_{j=0}^{n-x}\frac{(-1)^j}{j!}=1+\sum_{k=1}^{n}\frac{(z-1)^k}{k!}\,. \quad \textbf{(S10)}$$

In particular, $\sum_{x=0}^{n}P(x,n)=\sum_{x=0}^{n}\frac{1}{x!}\sum_{j=0}^{n-x}\frac{(-1)^j}{j!}=G((1,n)=1+\sum_{k=1}^{n}\frac{(1-1)^k}{k!}=1$, so Equation **(S7)** defines a valid probability distribution. The first and second moments of this distribution are defined as

$$\mu_x=\langle x\rangle=\sum_{x=0}^{n}xP(x,n)=\sum_{x=0}^{n}\frac{x}{x!}\sum_{j=0}^{n-x}\frac{(-1)^j}{j!}$$

$$\langle x^2\rangle=\sum_{x=0}^{n}x^2P(x,n)=\sum_{x=0}^{n}\frac{x^2}{x!}\sum_{j=0}^{n-x}\frac{(-1)^j}{j!}\,.$$

Now, $\frac{\partial G}{\partial z}=G_z(z,n)=\sum_{x=0}^{n}\frac{xz^{x-1}}{x!}\sum_{j=0}^{n-x}\frac{(-1)^j}{j!}=\sum_{k=1}^{n}\frac{k(z-1)^{k-1}}{k!}=1+\sum_{k=2}^{n}\frac{k(z-1)^{k-1}}{k!}$, so for any whole number *n*, $\mu_x=\langle x\rangle=G_z(1,n)=1$.

Similarly,

$$\frac{\partial^2 G}{\partial z^2} = G_{zz}(z,n) = \begin{cases} \displaystyle\sum_{x=0}^{n} \frac{x(x-1)z^{x-2}}{x!} \sum_{j=0}^{n-x} \frac{(-1)^j}{j!} = \sum_{k=2}^{n} \frac{k(k-1)(z-1)^{k-2}}{k!} = 1 + \sum_{k=3}^{n} \frac{k(z-1)^{k-1}}{k!} & \text{if } n \ge 2 \\ 0 & \text{if } n = 1 \end{cases}$$

so, $\left\langle x^2 \right\rangle = \displaystyle\sum_{x=0}^{n} x(x-1)P(x,n) + \sum_{x=0}^{n} xP(x,n) = G_{zz}(1,n) + \langle x \rangle = \begin{cases} 1+1=2 & \text{if } n \ge 2 \\ 0+1=1 & \text{if } n=1 \end{cases}$.

The variance of the distribution is given by $\sigma_x^2 = \left\langle x^2 \right\rangle - \langle x \rangle^2 = \begin{cases} 2-1^2=1 & \text{if } n \ge 2 \\ 1-1^2=0 & \text{if } n=1 \end{cases}$.

Note 5: Regarless of how large *n* is, if *n* > 1 , the average number of matches is always 1 with a standard deviation of 1. Surprisingly, these results **do not scale** with the number of objects considered. On a matching exam (i.e., *n* questions each with one and only one answer taken from a list of *n* items) a person randomly guessing answers scores no higher (on the average) on an exam with a hundred questions than on an exam with ten questions.