For a binomial distribution if *n* the number of trials is very large but the mean *np* stays roughly constant (meaning that the probability, *p* becomes very small) we have events which are relatively rare, but still occur with some regularity. For example, traffic accidents looked over time at a particular intersection or defects produced in a continuous output stream. The resulting distribution is a special limit of the binomial distribution and is named after the French mathematician, Simeon Poisson. A binomial distribution with *n* trials and a probability of success, *p*, on any one trial gives the following probability function for *x* successes in the *n* attempts,

$$P(x)=\binom{n}{x}p^x(1-p)^{n-x}.$$

Taking logarithms,

$$\ln\left[x!P(x)\right]=\ln(n!)-\ln\left[(n-x)!\right]+x\ln(p)+(n-x)\ln(1-p)$$

Replacing p by $p=\frac{\langle x\rangle}{n}=\mu$, gives

$$\ln\left[x!P(x)\right]=\ln(n!)-\ln\left[(n-x)!\right]+x\ln(\mu)-x\ln(n)+(n-x)\ln\left(1-\frac{\mu}{n}\right)$$

Stirling's Formula gives the asymptotic form of $n!$, $n!\approx\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$,so that

$$\ln(n!)\approx n\ln(n)-n\ln(e)+\frac{1}{2}\ln(n)+\frac{1}{2}\ln(2\pi)=n\ln(n)-n+\frac{\ln(n)}{2}+\frac{\ln(2\pi)}{2}$$

$$\ln\left[(n-x)!\right]\approx(n-x)\ln(n-x)-(n-x)+\frac{\ln(n-x)}{2}+\frac{\ln(2\pi)}{2}$$

$$\approx n\left(1-\frac{x}{n}\right)\ln\left[n\left(1-\frac{x}{n}\right)\right]-n+x+\frac{\ln\left[n\left(1-\frac{x}{n}\right)\right]}{2}+\frac{\ln(2\pi)}{2}$$

$$\approx n\left(1-\frac{x}{n}\right)\ln(n)+n\left(1-\frac{x}{n}\right)\ln\left(1-\frac{x}{n}\right)-n+x+\frac{\ln(n)}{2}+\frac{\ln\left(1-\frac{x}{n}\right)}{2}+\frac{\ln(2\pi)}{2}$$

The Maclaurin series for $\ln(1+q)=q-\frac{q^2}{2}+\frac{q^3}{3}-\frac{q^4}{4}+\frac{q^5}{5}\cdots$ converges for $|q|<1$.

Keeping terms to order $\frac{1}{n}$,

$$\ln\left[(n-x)!\right]\approx n\ln(n)-n+\frac{\ln(n)}{2}-x\ln(n)+x-x+\frac{\ln(2\pi)}{2}+\frac{x^2}{n}-\frac{x}{2n}$$, so that

$$\approx n\ln(n)-n+\frac{\ln(n)}{2}-x\ln(n)+\frac{\ln(2\pi)}{2}+\frac{x^2}{n}-\frac{x}{2n}$$

$$\ln(n!)-\ln\left[(n-x)!\right]\approx n\ln(n)-n+\frac{\ln(n)}{2}+\frac{\ln(2\pi)}{2}-n\ln(n)+n-\frac{\ln(n)}{2}+x\ln(n)-\frac{\ln(2\pi)}{2}-\frac{x^2}{n}+\frac{x}{2n}$$

$$\approx x\ln(n)-\frac{x^2}{n}+\frac{x}{2n}$$

Using this result in the expansion of $\ln\left[x!P(x)\right]$ gives

$$\begin{aligned}\ln\left[x!P(x)\right] &\approx x\ln(n)-\frac{x^2}{n}+\frac{x}{2n}+x\ln(\mu)-x\ln(n)+(n-x)\ln\left(1-\frac{\mu}{n}\right)\\ &\approx x\ln(\mu)+n\left(1-\frac{x}{n}\right)\ln\left(1-\frac{\mu}{n}\right)-\frac{x^2}{n}+\frac{x}{2n}\\ &\approx x\ln(\mu)-\mu\left(1-\frac{x}{n}\right)-\frac{x^2}{n}+\frac{x}{2n}=x\ln(\mu)-\mu+O\left(\frac{1}{n}\right)\\ &\approx \ln\left(\mu^x\right)+\ln\left(e^{-\mu}\right)+O\left(\frac{1}{n}\right)=\ln\left(e^{-\mu}\mu^x\right)+O\left(\frac{1}{n}\right)\end{aligned}$$

So as $n\to\infty$, $\ln\left[x!P(x)\right]\to\ln\left(e^{-\mu}\mu^x\right)$, or $x!P(x)\to e^{-\mu}\mu^x$. This yields the Poisson probability distribution.

$$P(x)=\frac{e^{-\mu}\mu^x}{x!}\ . \qquad \textbf{(1)}$$

Since $n\to\infty$, x can be any whole number. To see that this defines a valid probability distribution, use the Maclaurin series for the exponential function.

$$\sum_{x=0}^{\infty}P(x)=\sum_{x=0}^{\infty}\frac{e^{-\mu}\mu^x}{x!}=e^{-\mu}\sum_{x=0}^{\infty}\frac{\mu^x}{x!}=e^{-\mu}e^{\mu}=1$$

The mean of the Poisson distribution is calculated as follows:

$$\begin{aligned}\mu_x=\langle x\rangle &=\sum_{x=0}^{\infty}xP(x)=\sum_{x=0}^{\infty}\frac{xe^{-\mu}\mu^x}{x!}=e^{-\mu}\sum_{x=1}^{\infty}\frac{x\mu^x}{x(x-1)!}=e^{-\mu}\sum_{x=1}^{\infty}\frac{\mu^x}{(x-1)!}\\ &=e^{-\mu}\sum_{x=1}^{\infty}\frac{\mu^{x-1}\mu^1}{(x-1)!}=\mu e^{-\mu}\sum_{x=1}^{\infty}\frac{\mu^{x-1}}{(x-1)!}=\mu e^{-\mu}\sum_{x=0}^{\infty}\frac{\mu^x}{x!}=\mu e^{-\mu}e^{\mu}=\mu\end{aligned}$$

Thus, the mean is just μ. But this was to be expected! We started with a binomial distribution of mean μ and it wasn't supposed to change as n increased. To calculate the variance we calculate the second moment about the origin.

$$\left\langle x^2\right\rangle=\left\langle x^2-x\right\rangle+\langle x\rangle=\left\langle x(x-1)\right\rangle+\mu$$

$$\begin{aligned}\left\langle x(x-1)\right\rangle &=\sum_{x=0}^{\infty}x(x-1)P(x)=\sum_{x=0}^{\infty}\frac{x(x-1)e^{-\mu}\mu^x}{x!}=e^{-\mu}\sum_{x=2}^{\infty}\frac{x(x-1)\mu^x}{x(x-1)(x-2)!}=e^{-\mu}\sum_{x=2}^{\infty}\frac{\mu^x}{(x-2)!}\\ &=e^{-\mu}\sum_{x=2}^{\infty}\frac{\mu^{x-2}\mu^2}{(x-2)!}=\mu^2e^{-\mu}\sum_{x=2}^{\infty}\frac{\mu^{x-2}}{(x-2)!}=\mu^2e^{-\mu}\sum_{x=0}^{\infty}\frac{\mu^x}{x!}=\mu^2e^{-\mu}e^{\mu}=\mu^2\end{aligned}$$

So, $\left\langle x^2\right\rangle=\mu^2+\mu$; $\sigma_x^2=\left\langle x^2\right\rangle-\langle x\rangle^2=\mu^2+\mu-\mu^2=\mu$. The variance of a Poisson distribution is the same as its mean.