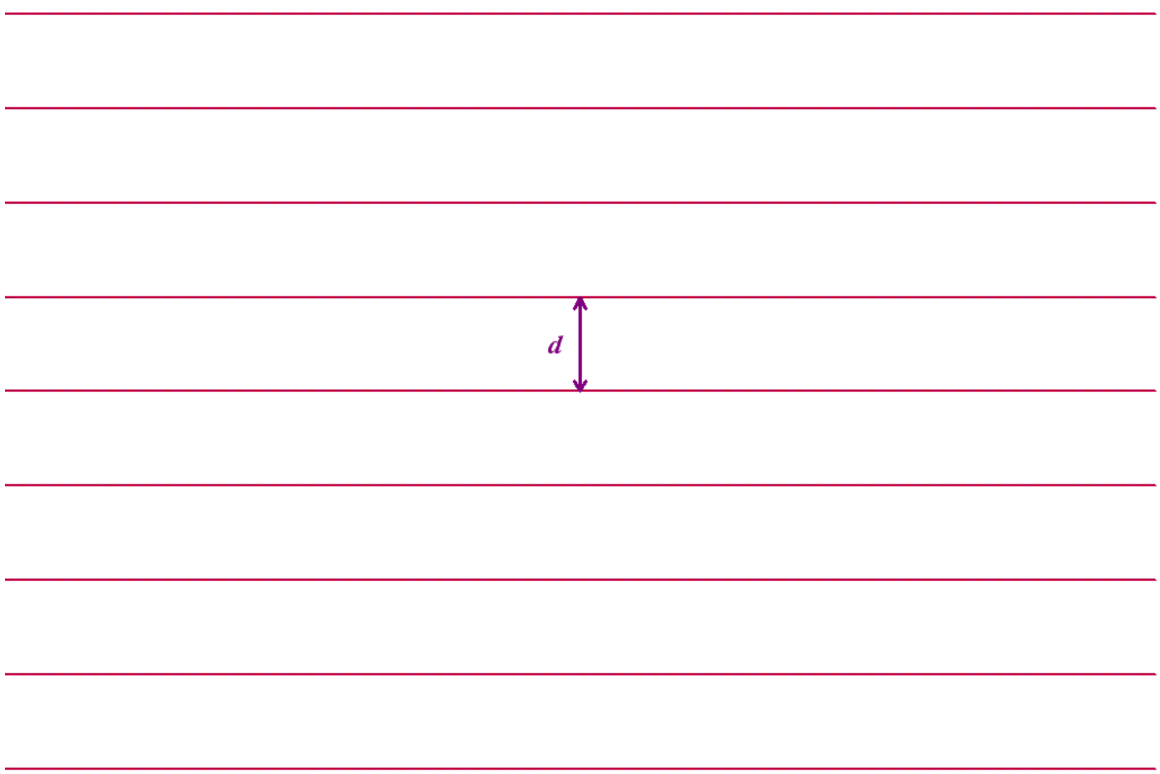
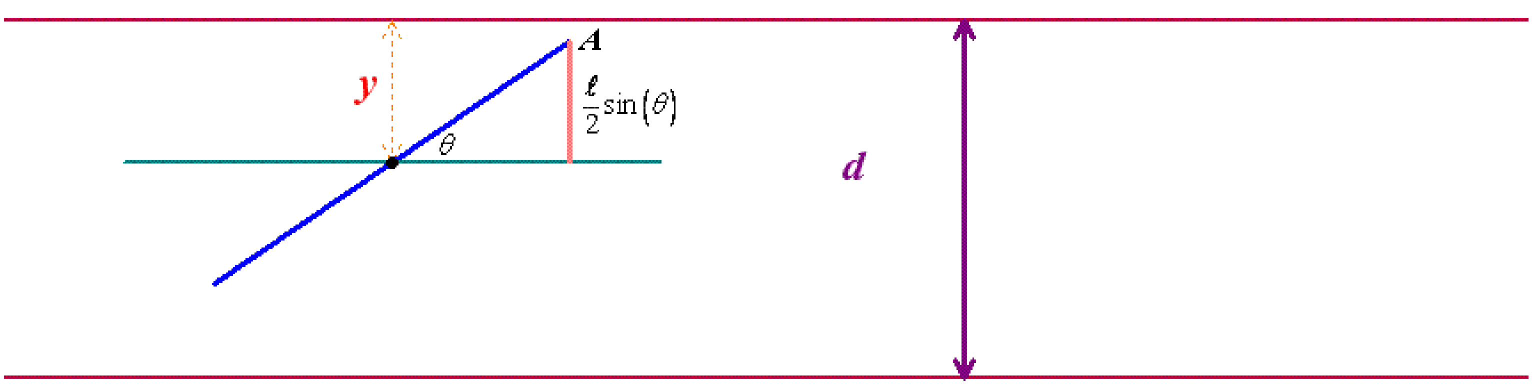
Consider a set of parallel lines each separated by a distance *d*. Now imagine "throwing" a



pin or needle of length ℓ on this surface. If the throw randomly positions the pin, what is the probability that the pin touches at least one of the parallel lines? This was one of the first and most famous problems in what is called Geometric Probability. It was first proposed and solved by Georges-Louis Leclerc, Comte de Buffon in 1777. To analyze this problem let the random variable *y* be the closest length of the center of the pin to one of the parallel lines. Construct a line parallel to the set of parallel lines though the center of the pin. Let ***A*** label the end of the pin between this line and the closest of the parallel lines to the pin's center. Let the random variable θ be the angle in $[0,\pi]$ that the segment from the center of the pin to ***A*** makes with the parallel lines.



Under the reasonable assumption that the toss gives *y* independent of θ, the joint probability distribution density function is given by the following product of two uniform distributions

$$f(y,\theta) = f_y(y)\, f_\theta(\theta) = \begin{cases} \dfrac{1}{d/2}\dfrac{1}{\pi} = \dfrac{2}{d\pi} & \text{if } 0<\theta<\pi \text{ and } 0<y<\dfrac{d}{2} \\ 0 & \text{elsewhere} \end{cases} .$$

Now, the segment from the center of the pin to *A* will cross one of the parallel lines if and only if $y \le \frac{\ell}{2}\sin(\theta)$. For the case when $\ell > d$, this will always be true for any θ in the interval $\left[\sin^{-1}\left(\frac{d}{\ell}\right), \pi - \sin^{-1}\left(\frac{d}{\ell}\right)\right]$ since then $y \le \frac{d}{2}$. Thus, the probability that a pin with a longer length than the separation between the parallel lines touches at least one of the lines is given by the sum of the following three intgrals.

$$P(d,\ell;d<\ell) = \int_0^{\sin^{-1}\left(\frac{d}{\ell}\right)} \int_0^{\ell/2\sin(\theta)} f(y,\theta)\,dy\,d\theta + \int_{\sin^{-1}\left(\frac{d}{\ell}\right)}^{\pi-\sin^{-1}\left(\frac{d}{\ell}\right)} \int_0^{d/2} f(y,\theta)\,dy\,d\theta + \int_{\pi-\sin^{-1}\left(\frac{d}{\ell}\right)}^{\pi} \int_0^{\ell/2\sin(\theta)} f(y,\theta)\,dy\,d\theta .$$

Now,

$$\int_0^{\sin^{-1}\left(\frac{d}{\ell}\right)} \int_0^{\ell/2\sin(\theta)} \frac{2}{d\pi}\,dy\,d\theta = \frac{2}{d\pi}\int_0^{\sin^{-1}\left(\frac{d}{\ell}\right)} \frac{\ell}{2}\sin(\theta)\,d\theta$$

$$= \frac{\ell}{d\pi}\Big[-\cos(\theta)\Big]_0^{\sin^{-1}\left(\frac{d}{\ell}\right)} = \frac{\ell}{d\pi}\left[1-\cos\left(\sin^{-1}\left(\frac{d}{\ell}\right)\right)\right]$$

$$\frac{\ell}{d\pi}\left[1-\sqrt{1-d^2/\ell^2}\right]$$

and

$$\int_{\pi-\sin^{-1}\left(\frac{d}{\ell}\right)}^{\pi} \int_0^{\ell/2\sin(\theta)} \frac{2}{d\pi}\,dy\,d\theta = \frac{2}{d\pi}\int_{\pi-\sin^{-1}\left(\frac{d}{\ell}\right)}^{\pi} \frac{\ell}{2}\sin(\theta)\,d\theta = \frac{\ell}{d\pi}\int_{\pi-\sin^{-1}\left(\frac{d}{\ell}\right)}^{\pi} \sin(\theta)\,d\theta$$

$$= \frac{\ell}{d\pi}\Big[-\cos(\theta)\Big]_{\pi-\sin^{-1}\left(\frac{d}{\ell}\right)}^{\pi} = \frac{\ell}{d\pi}\left[1+\cos\left(\pi-\sin^{-1}\left(\frac{d}{\ell}\right)\right)\right],$$

$$= \frac{\ell}{d\pi}\left[1-\cos\left(\sin^{-1}\left(\frac{d}{\ell}\right)\right)\right] = \frac{\ell}{d\pi}\left[1-\sqrt{1-d^2/\ell^2}\right]$$

while

$$\int_{\sin^{-1}\left(\frac{d}{\ell}\right)}^{\pi-\sin^{-1}\left(\frac{d}{\ell}\right)} \int_0^{d/2} f(y,\theta)\,dy\,d\theta = \int_{\sin^{-1}\left(\frac{d}{\ell}\right)}^{\pi-\sin^{-1}\left(\frac{d}{\ell}\right)} \int_0^{d/2} \frac{2}{d\pi}\,dy\,d\theta$$

$$= \frac{1}{\pi}\int_{\sin^{-1}\left(\frac{d}{\ell}\right)}^{\pi-\sin^{-1}\left(\frac{d}{\ell}\right)} d\theta = \frac{1}{\pi}\left[\pi - 2\sin^{-1}\left(\frac{d}{\ell}\right)\right].$$

$$= 1 - \frac{2}{\pi}\sin^{-1}\left(\frac{d}{\ell}\right)$$

Hence, $P(d,\ell;d<\ell) = 1 + \frac{2\ell}{d\pi}\left[1-\sqrt{1-d^2/\ell^2}\right] - \frac{2}{\pi}\sin^{-1}\left(\frac{d}{l}\right)$, note, as would be expected, as ℓ becomes much larger than *d*, this probability goes to one. This can be demonstrated using Maclaurin series as follows:

$$P(d,\ell;d<\ell)=1+\frac{2\ell}{d\pi}\left[1-\left(1-d^2/\ell^2\right)^{1/2}\right]-\frac{2}{\pi}\sin^{-1}\left(\frac{d}{l}\right)=1+\frac{2\ell}{d\pi}\left[1-\left(1-\frac{d^2}{2\ell^2}+O\left(\frac{d^4}{\ell^4}\right)\right)\right]-\frac{2}{\pi}\left(\frac{d}{l}\right)+O\left(\frac{d^2}{\ell^2}\right)$$
$$=1+\frac{2\ell}{d\pi}\left(\frac{d^2}{2\ell^2}\right)-\frac{2}{\pi}\left(\frac{d}{l}\right)+O\left(\frac{d^2}{\ell^2}\right)=1-\frac{1}{\pi}\left(\frac{d}{l}\right)+O\left(\frac{d^2}{\ell^2}\right).$$

The simpler case is when $d \ge \ell$, then $y \le \frac{\ell}{2}\sin(\theta)$ is always less than $\frac{d}{2}$,

$$P(d,\ell;d\ge\ell)=\int_0^{\pi}\int_0^{\ell/2\sin(\theta)} f(y,\theta)\,dy\,d\theta=\frac{2\ell}{2d\pi}\int_0^{\pi}\sin(\theta)\,d\theta$$
$$=\frac{\ell}{d\pi}\left[-\cos(\theta)\right]_0^{\pi}=\frac{2\ell}{d\pi}$$

For the special case of $d=\ell$ (i.e., the spacing exactly matches the length of the needle), the probability that the pin crosses a line is simply given by $P(d,d;d=\ell)=\frac{2}{\pi}=0.636619772\ldots$.

In 1901 Mario Lazzarini claimed to have tossed pins which had a ratio of $\frac{\ell}{d}=\frac{5}{6}$. He stated that the pins touched a line 1808 out of the 3408 times he did the experiment. The theoretical probability for this value of $\frac{\ell}{d}$ is $\frac{2\ell}{d\pi}=\frac{2(5)}{6\pi}=\frac{5}{3\pi}$. The observed proportion (empirical probability) is $\frac{1808}{3408}=\frac{113}{213}$. Using this as an "experimental estimate" of Pi gives $\frac{5}{3\text{Pi}}=\frac{113}{213}\rightarrow\frac{3\text{Pi}}{5}=\frac{213}{113}\rightarrow\text{Pi}=\frac{5}{3}\left(\frac{213}{113}\right)=\frac{(5)71}{113}=\frac{355}{113}=3.14159292$. This is a percent error of $8.49\times10^{-6}\%$! How credible is this claim? Using a binomial distribution with n = 3408 and $p=\frac{5}{3\pi}$, the expected number of times the pin would cross a line is given by $np=(3408)\frac{5}{3\pi}=1808.000154$. This distribution has a standard deviation of $\sqrt{np(1-p)}=\sqrt{3408\frac{5}{3\pi}\left(1-\frac{5}{3\pi}\right)}=29.13462342$. What is the probability that you would get x = 1808? Using a standard normal distribution to approximate the large n binomial distribution and correcting for continuity gives

$$P_{\text{SN}}\left(-\frac{0.500154}{29.13462342}<z<\frac{0.499846}{29.13462342}\right)=P_{\text{SN}}\left(-0.0171669809<z<0.01756422\right)$$
$$=0.013692$$

So either Lazzarini was pretty lucky, or he "fudged" his data. I suspect the latter!

Other interesting problems in Geometric Probability can be found at

http://faculty.matcmadison.edu/alehnen/sphere/prob.htm and

http://faculty.matcmadison.edu/alehnen/sphere/hypers.htm .