Bonus Project 1 **A Deranged Experiment** /15

Name ________________________ **Due 3/12/2009**

Consider *n* letters with *n* different address headings and *n* letters each addressed to one of these addresses. Suppose before the letters are stuffed into the envelopes, they are dropped on the floor and then carelessly (randomly) assigned to the *n* envelopes. Another statement of essentially the same problem is as follows: *n* men check in their hats at a club. Due to employee shortages at the club, as the men leave they are randomly given a hat from the collection of *n* hats. A more dramatic but equivalent scenario might have *n* patients' medical records and *n* lab results. Because of some mix up, the tests results are randomly assigned to the medical records. An assignment of the *n* objects (letters, test results, etc.) to *n* receptacles (envelopes, lab reports, etc.) such that **none** of them is in the proper position is called a derangement of the *n* objects. A more abstract algebraic definition of a derangement of *n* objects is a permutation of order *n* which has no fixed points. Over two hundred fifty years ago Leonhard Euler derived a formula for the number of different derangements of *n* objects in the context of analyzing the probability of winning a card game called Rencontre.

The Experiment:

Write the numbers 1 through 16 on 16 identical (as close as you can make them) slips of paper. Be sure to distinguish a 6 from a 9 . Put these slips into an envelope, stir them up and randomly draw out each slip, one at a time. As each slip is drawn, place it on a marked four by four grid such as the one shown below. The first slip goes in the cell marked 1, the second in the cell marked 2, and so on. Record how many of the slip numbers match the grid number of the slip. The number of "correct" matches, *x*, is the relevant random variable.

1	**2**	**3**	**4**
5	**6**	**7**	**8**
9	**10**	**11**	**12**
13	**14**	**15**	**16**

Repeat the experiment 100 times and record your results in Table 1.

The Analysis

In order to determine the number of derangements and the probability distribution of *x*, consider the following line of thought. Let $C(x,n)$ be the number of distinct ways that there are *x* correct matches out of *n*. For convenience adopt the definition that $C(0,0) \equiv 1$. The following values should be "obvious"; fill them in.

A. $C(n,n) =$ ___________ $C(0,1) =$ ___________ $C(n-1,n) =$ ___________

Table 1

Event	**Observed *f***	**Empirical Probability**	**Theoretical Probability**
$x = 0$			
$x = 1$			
$x = 2$			
$x > 2$			
	μ_x		
	σ_x		

To determine a recursion for $C(0, n)$, let $E(n)$ designate the number of distinct one-to-one functions, f_i, from the set $\{e_1, e_2, e_3, \ldots e_{n-1}, q\}$ onto the set $\{e_1, e_2, e_3, \ldots e_{n-1}, e_n\}$ where $q \neq e_n$ and $f_i(s) \neq s$ for s in $\{e_1, e_2, e_3, \ldots e_{n-1}\}$. Since there is only one function from $\{q\}$ onto $\{e_1\}$, $E(1) = 1$. Similarly, there is only one function from $\{e_1, q\}$ onto $\{e_1, e_2\}$, for which $f(s) \neq s$, namely, $f(e_1) = e_2$, and $f(q) = e_1$. So, $E(2) = 1$. As is shown below there are exactly three functions from $\{e_1, e_2, q\}$ onto $\{e_1, e_2, e_3\}$ with $f_i(s) \neq s$. So, $E(3) = 3$.

$f_1(q) = e_1$	$f_2(q) = e_2$	$f_3(q) = e_3$
$f_1(e_1) = e_2$	$f_2(e_1) = e_3$	$f_3(e_1) = e_2$
$f_1(e_2) = e_3$	$f_2(e_2) = e_1$	$f_3(e_2) = e_1$

B. Develop a recursion that expresses $E(n)$ in terms of $C(0, n-1)$ and $E(n-1)$.

C. Develop a recursion that expresses $C(0, n)$ in terms of and $E(n-1)$.

D. From the recursions in B. and C., develop a recursion that expresses $C(0, n)$ in terms of $C(0, n-1)$ and $C(0, n-2)$.

E. Use your recursion to fill in the following.

$C(0,0) =$ ____1____ $C(0,1) =$ ____0____ $C(0,2) =$ _________ $C(0,3) =$ _________

$C(0,4) =$ _________ $C(0,5) =$ _________ $C(0,6) =$ _________ $C(0,7) =$ _________

F. Using the results from E., guess a formula for $C(0,n) - nC(0,n-1)$. Verify your guess using mathematical induction.

G. $C(0,n)$ can be expressed as the sum, $C(0,n) = n!\sum_{j=0}^{n} a_j$. Using the result from F., determine an explicit formula for a_j .

H. Using the result from G., evaluate $\lim_{n\to\infty} \frac{C(0,n)}{n!}$.

I. Using the result from G. and the properties of binomial coefficients, give an explicit formula for $P(x,n)$, the probability of x correct matches out of n.

J. Using the probability derived in I., calculate the moments $\langle x \rangle$ and $\langle x^2 \rangle$ and from them the mean, μ_x, and standard deviation, σ_x, of the theoretical probability distribution.

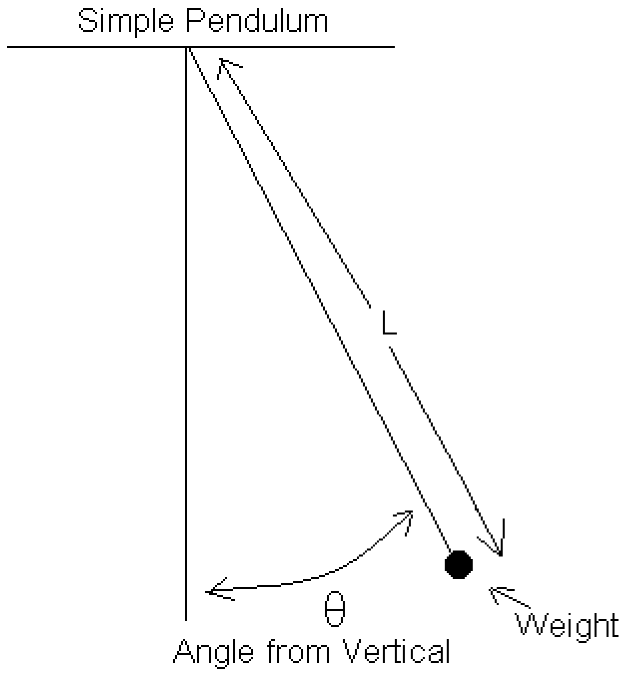
K. In Table 1, how do the empirical and theoretical event probabilities compare? How do the empirical and theoretical means compare? How do the empirical and theoretical standard deviations compare?

Bonus Project 2 **The Simple Pendulum** /15

Name ________________________ **Due At Final Exam**

Building the model:

The ideal simple pendulum consists of a weight of mass, *m*, suspended by a massless string of length *L*. The angular displacement of the mass from the vertical is designated by θ .



It is assumed that the mass is released from rest at an initial angle of θ_0 .Ignoring all friction and using Newton's Second Law of Motion, one can describe the pendulum's dynamics with the following second order ODE.

$$mL\frac{d^2\theta}{dt^2} = -mg\sin(\theta)$$

Here g is the acceleration of gravity. The initial conditions are that at $t = 0$, $\theta = \theta_0$ and $\frac{d\theta}{dt} = 0$. Since the mass divides out of both sides (The equivalence of inertial and gravitational mass!), the only independent variables of the model are L and θ_0 with g as a parameter. The small angle approximation, which even for an angle as large as $30° = \frac{\pi}{6}$ still gives results which are within 2% of the exact solution, replaces $\sin(\theta)$ by θ , the first non-zero term in the Maclaurin series of $\sin(\theta)$. Using this approximation solve for θ as a function of t.

The period of the simple pendulum, T, is the time interval required for θ to return to its initial position. Solve for T in the context of the small angle approximation.

Testing the Model:
In the table below is data collected by a group of students in a Technical Math Class. The first mass was a steel ball, the second was a ball made of wood. Using a meter stick the length of the pendulum was set to within 1 mm of the values shown. Periods were obtained by using a stop watch to record 10 full swings at the specified L and initial angle; this time was then divided by 10.

	Mass = 66.1g		**Mass = 5.8g**	
	$\theta_0 = 20°$	$\theta_0 = 30°$	$\theta_0 = 20°$	$\theta_0 = 30°$
***L* (cm)**	**Period *T* (sec)**	**Period *T* (sec)**	**Period *T* (sec)**	**Period *T* (sec)**
10	0.614	0.660	0.615	0.631
15	0.784	0.791	0.753	0.756
20	0.894	0.932	0.881	0.878
25	1.019	1.016	0.987	1.003
30	1.115	1.118	1.090	1.060
35	1.213	1.213	1.159	1.191
40	1.287	1.284	1.253	1.231
45	1.366	1.350	1.309	1.347
50	1.466	1.409	1.393	1.404
55	1.422	1.494	1.475	1.415
60	1.597	1.501	1.537	1.537
65	1.609	1.603	1.568	1.622
70	1.625	1.669	1.668	1.688
75	1.825	1.744	1.753	1.716
80	1.821	1.793	1.766	1.788

A. In the small angle approximation what dependence does the model predict for the period in terms of the initial angle?

Does this data support that result? Describe in detail how you used the data to answer the question. Perform all relevant calculations in Excel and attach your spreadsheet with the assignment.

B. What dependence does the model predict for the period in terms of the mass of the ball?

Does this data support that result? Describe in detail how you used the data to answer the question. Perform all relevant calculations in Excel and attach your spreadsheet with the assignment.

C. Use the data to estimate the model's lone parameter, g. Describe in detail how you used the data to do the estimate. Perform all relevant calculations in Excel and attach your spreadsheet with the assignment.

Bonus Project 3 **Two Factor Analysis** /15

Name ________________________ **Due At Final Exam**

This problem is based on the first major project I did as an engineer at Ray-O-Vac.

Lithium (Li) thionyl choride ($SOCl_2$) batteries offer a great deal of advantages due to their superior energy density (voltage/mass). A problem, however, was the formation of LiCl at the lithium anode, which while it prevents the non-electrical consumption of the lithium, also introduced a "voltage delay" when the battery was discharged across a load. By accident it was discovered that a polymer of cyanoacrylate ("super glue") help in mitigating this effect. A second treatment was to subject each battery after assembly to voltage under load (VUL) test by allowing it to discharge across a 50 ohm resistor for 10 seconds. This procedure was rather cumbersome so it would be difficult to implement in a full scale manufacturing setting. In addition, the amount and placement of the cyanoacrylate within the battery for optimal performance needed to be investigated. In this regard, 16 different pilot assembly runs were conducted. There were 8 different designs of cyanoacrylate amount and application and each assembly was replicated. The different designs were randomly assigned to each pilot assembly. For each batch of batteries produced, half received VUL and half did not. Two response variables of interest were the open circuit voltage (OCV) measured with a potentiometer and the battery capacity. The latter was the total charge delivered in amp hours when the battery was discharged across a 100 ohm load until a 2.7 end point voltage is attained. The results for both response variables are given in the following tables.

OCV	**Replication 1**		**Replication 2**	
Design	**VUL**	**No VUL**	**VUL**	**No VUL**
1	3.623	3.626	3.626	3.632
	3.623	3.632	3.626	3.626
	3.623	3.626	3.626	3.626
2	3.618	3.619	3.619	3.622
	3.615	3.619	3.622	3.626
	3.618	3.619	3.622	3.626
3	3.622	3.618	3.622	3.626
	3.618	3.618	3.622	3.626
	3.619	3.618	3.622	3.626
4	3.622	3.634	3.626	3.634
	3.622	3.632	3.626	3.634
	3.626	3.632	3.626	3.632
5	3.618	3.618	3.622	3.622
	3.618	3.618	3.618	3.622
	3.618	3.622	3.622	3.626
6	3.622	3.622	3.626	3.626
	3.618	3.622	3.622	3.626
	3.618	3.622	3.626	3.626
7	3.622	3.622	3.622	3.622
	3.622	3.622	3.618	3.626
	3.622	3.622	3.618	3.622
8	3.622	3.622	3.622	3.626
	3.622	3.623	3.622	3.619
	3.622	3.626	3.618	3.619

Capacity	Replication 1		Replication 2	
Factor A	VUL	No VUL	VUL	No VUL
1	1.359	1.450	1.283	1.313
	1.473	1.489	1.313	1.271
	1.492	1.433	1.313	1.29
2	1.448	1.490	1.173	1.52
	1.506	1.493	1.511	1.506
	1.478	1.444	1.496	1.505
3	1.472	1.506	1.513	1.502
	1.468	1.519	1.509	1.525
	1.476	1.502	1.531	1.505
4	1.444	1.474	1.342	1.279
	1.464	1.480	1.291	1.268
	1.432	1.464	1.323	1.397
5	1.476	1.504	1.497	1.502
	1.478	1.480	1.508	1.527
	1.488	1.502	1.518	1.524
6	1.433	1.452	1.43	1.468
	1.349	1.392	1.465	1.408
	1.328	1.352	1.419	1.455
7	1.382	1.362	1.426	1.439
	1.357	1.418	1.420	1.337
	1.419	1.511	1.472	1.413
8	1.312	1.334	1.403	1.425
	1.288	1.287	1.442	1.400
	1.311	1.294	1.344	1.453

Perform an analysis which answers the following questions for both OCV and Capacity.

1. Is there a significant effect (alpha = 0.05) due to the design variations?
2. Is there a significant effect (alpha = 0.05) due to VUL?
3. Is there evidence (alpha = 0.05) an interaction between design variations and VUL?
4. Is there a significant effect (alpha = 0.05) due to replication?
5. Calculate a 95% confidence interval for the main effect of VUL.

Explain why the confidence interval calculation for VUL does NOT use the mean square associated with the VUL factor.