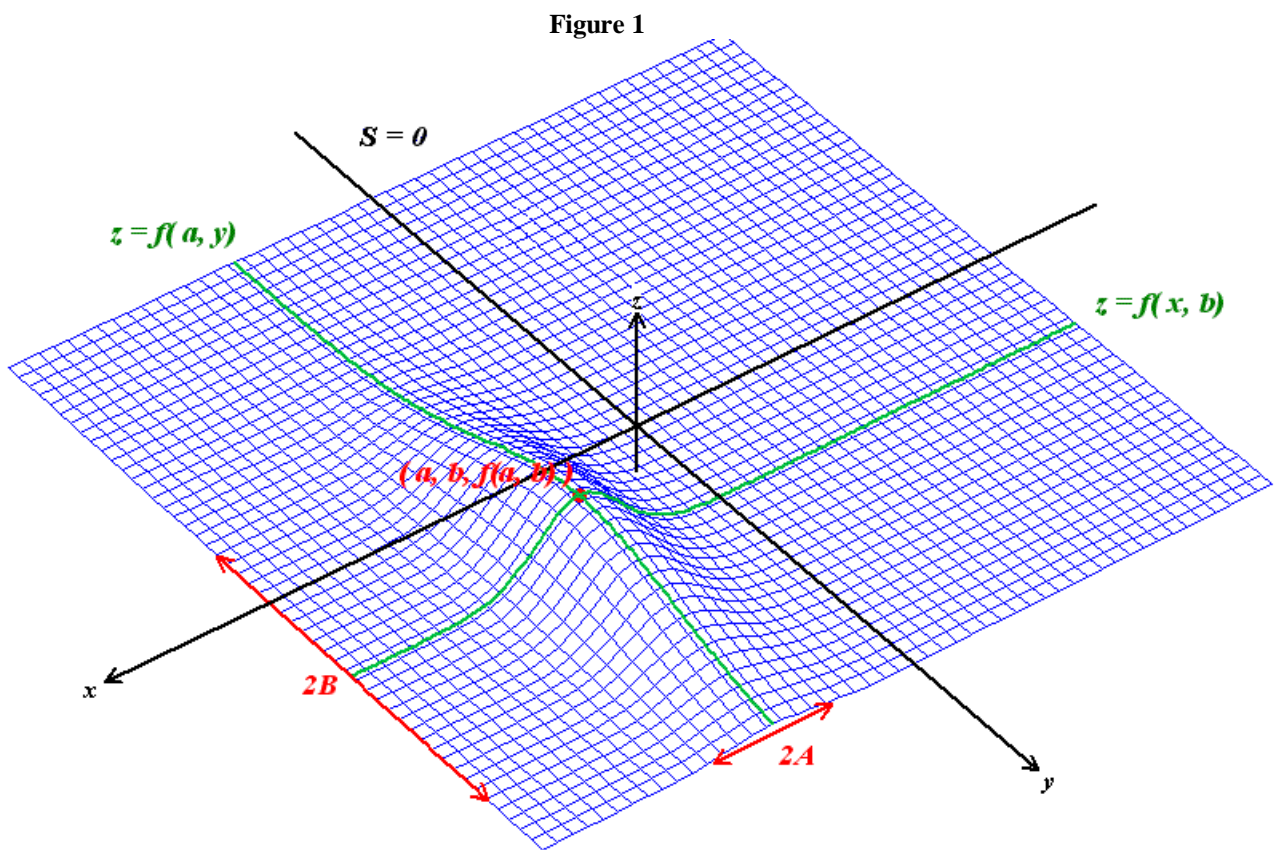


Properties of the Bivariate Normal Distribution

An important continuous joint probability distribution is the bivariate normal distribution. Furthermore, it provides an example that nicely illustrates the steps in the analysis of a joint probability distribution. Consider the following function of x and y for positive A and B and $-1 < S < 1$,

$$f(x, y) = \frac{1}{2\pi AB\sqrt{1-S^2}} \exp \left[-\frac{\left(\frac{x-a}{A}\right)^2 - 2S\left(\frac{x-a}{A}\right)\left(\frac{y-b}{B}\right) + \left(\frac{y-b}{B}\right)^2}{2(1-S^2)} \right]. \quad (1)$$

Graphs of this function are displayed Figures 1, 2 and 3 for three different values of S .



Now, is $f(x, y)$ a valid two variable continuous probability density function? Certainly $f(x, y) > 0$. The remaining condition to verify is that $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dy dx = 1$. Completing the square within the exponential yields

$\left(\frac{x-a}{A}\right)^2 - 2S\left(\frac{x-a}{A}\right)\left(\frac{y-b}{B}\right) = \left[\frac{x-a}{A} - S\left(\frac{y-b}{B}\right)\right]^2 - S^2\left(\frac{y-b}{B}\right)^2$. This leads to the following linear transformation that defines new variables u and v .

$$u = \frac{x-a}{A} - S\left(\frac{y-b}{B}\right) \quad v = \sqrt{1-S^2}\left(\frac{y-b}{B}\right) \quad (2)$$

Figure 2

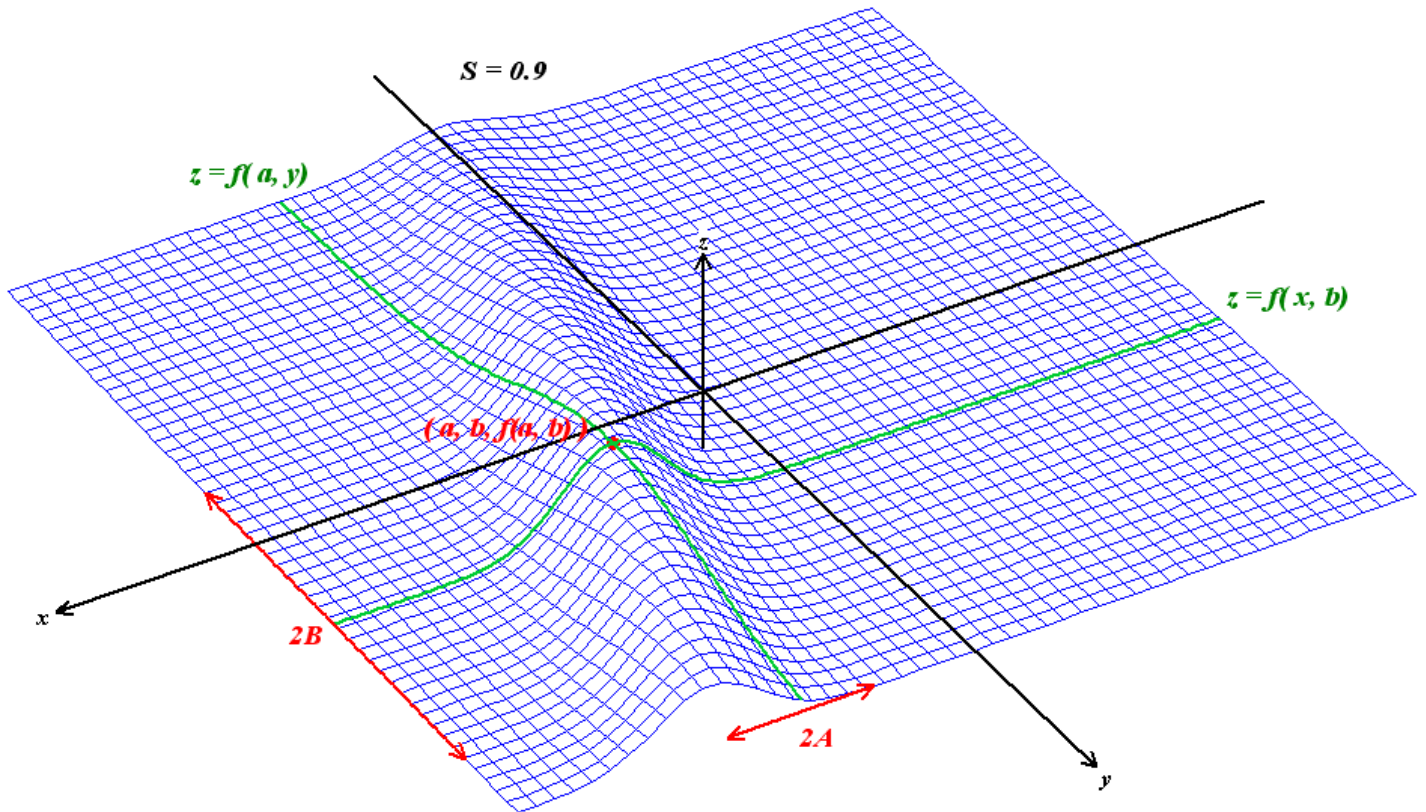
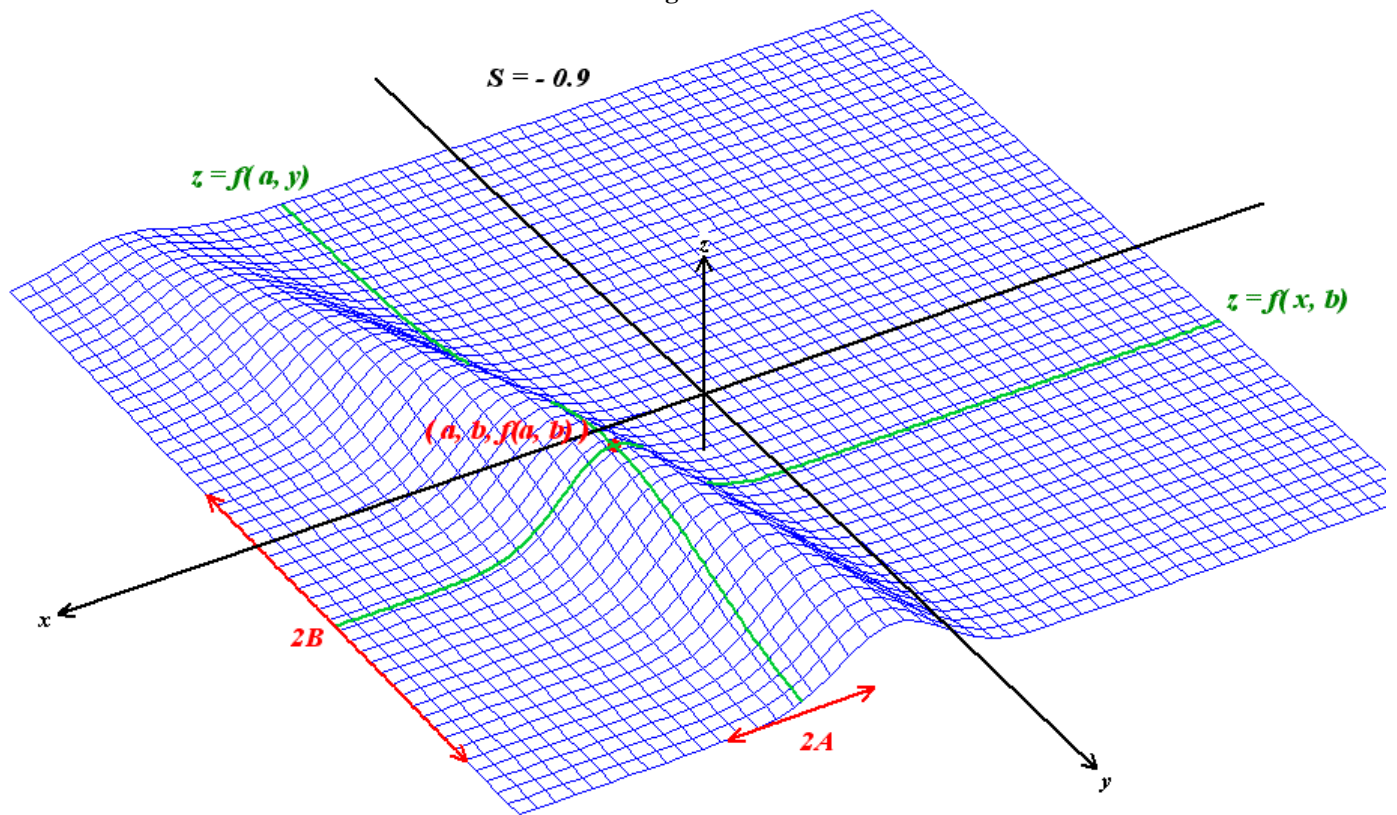


Figure 3



The inverse transformation is given by

$$x(u,v) = a + Au + \frac{SA}{\sqrt{1-S^2}}v \quad y(u,v) = b + \frac{B}{\sqrt{1-S^2}}v \quad (3)$$

and has the Jacobian $\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \begin{vmatrix} A & \frac{SA}{\sqrt{1-S^2}} \\ 0 & \frac{B}{\sqrt{1-S^2}} \end{vmatrix} = \frac{AB}{\sqrt{1-S^2}}$. In transforming the integral $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) dy dx$ the inverse

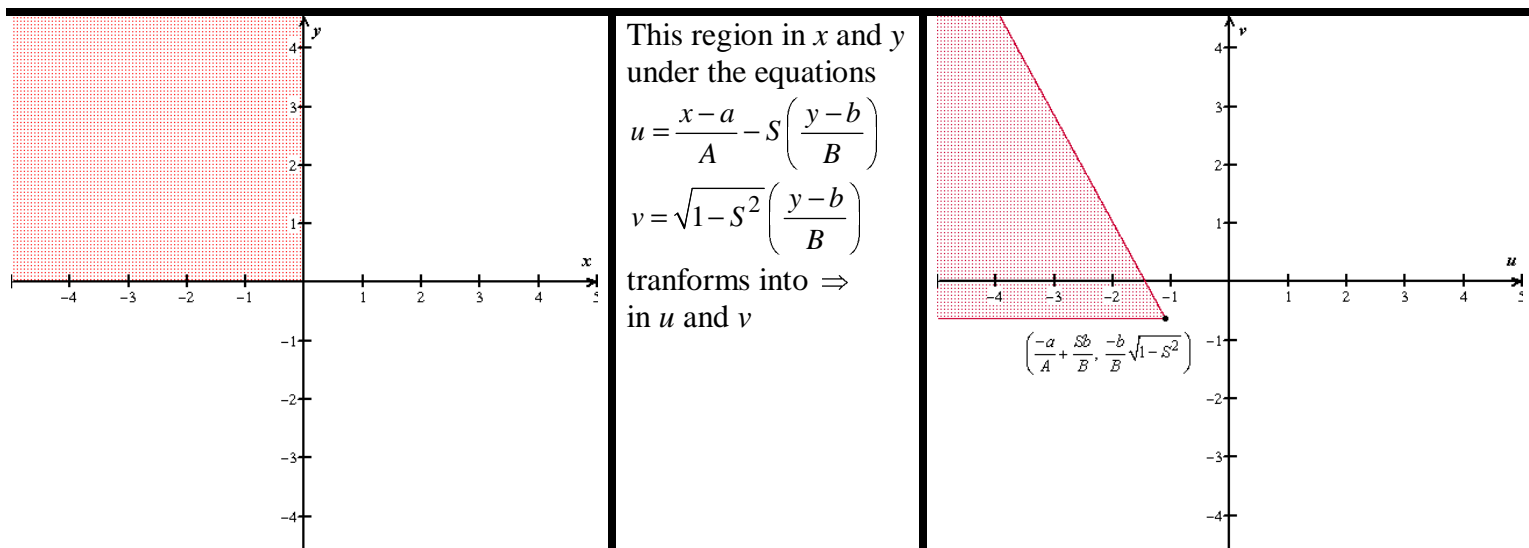
transformation is used to express the integrand into the following function of u and v ,

$$f(x(u,v), y(u,v)) = F(u,v) = \frac{1}{2\pi AB\sqrt{1-S^2}} \exp\left[-\frac{u^2 + v^2}{2(1-S^2)}\right]. \quad (4)$$

The direct transformation of equation (2) is used to transform the boundaries. The second quadrant boundaries in x, y are $-\infty < x < 0; y = 0$ and $x = 0; 0 < y < \infty$. In u, v these become $-\infty < u < \frac{Sb}{B} - \frac{a}{A}; v = -\frac{b}{B}\sqrt{1-S^2}$ and

$-\frac{b}{B}\sqrt{1-S^2} < v < \infty; u = \frac{-a}{A} - \frac{S}{\sqrt{1-S^2}}v$. This is illustrated below for $S > 0$. If $S < 0$, the upper boundary line in u, v

would have a positive rather than a negative slope. If $S = 0$, this boundary becomes a vertical line.



The first quadrant boundaries in x, y are $0 < x < \infty; y = 0$ and $x = 0; 0 < y < \infty$. In u, v these become

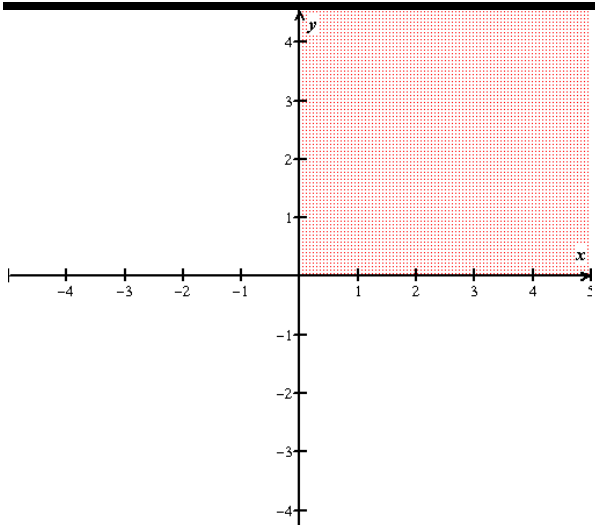
$\frac{Sb}{B} - \frac{a}{A} < u < \infty; v = -\frac{b}{B}\sqrt{1-S^2}$ and $-\frac{b}{B}\sqrt{1-S^2} < v < \infty; u = \frac{-a}{A} - \frac{S}{\sqrt{1-S^2}}v$. This is illustrated in the table below for

$S > 0$. If $S < 0$, the left boundary line in u, v would have a positive rather than a negative slope. If $S = 0$, this boundary becomes a vertical line.

The third quadrant boundaries in x, y are $-\infty < x < 0; y = 0$ and $x = 0; -\infty < y < 0$. In u, v these become

$-\infty < u < \frac{Sb}{B} - \frac{a}{A}; v = -\frac{b}{B}\sqrt{1-S^2}$ and $-\infty < v < -\frac{b}{B}\sqrt{1-S^2}; u = \frac{-a}{A} - \frac{S}{\sqrt{1-S^2}}v$. This is illustrated in the table for $S > 0$.

If $S < 0$, the right boundary line in u, v would have a positive rather than a negative slope. If $S = 0$, this boundary becomes a vertical line.

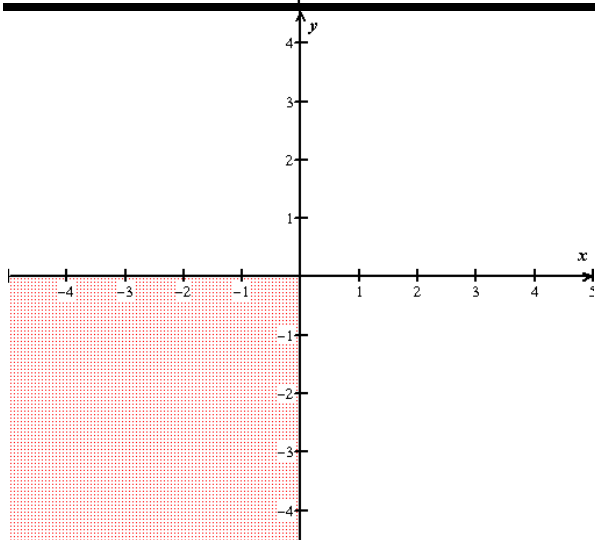
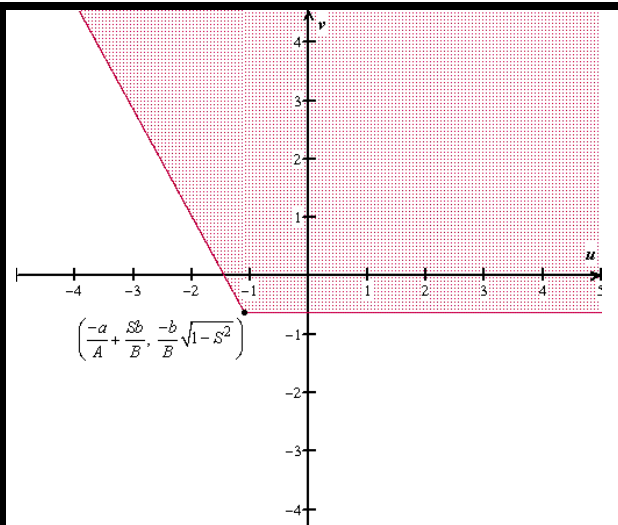


This region in x and y under the equations

$$u = \frac{x-a}{A} - S \left(\frac{y-b}{B} \right)$$

$$v = \sqrt{1-S^2} \left(\frac{y-b}{B} \right)$$

transforms into \Rightarrow
in u and v

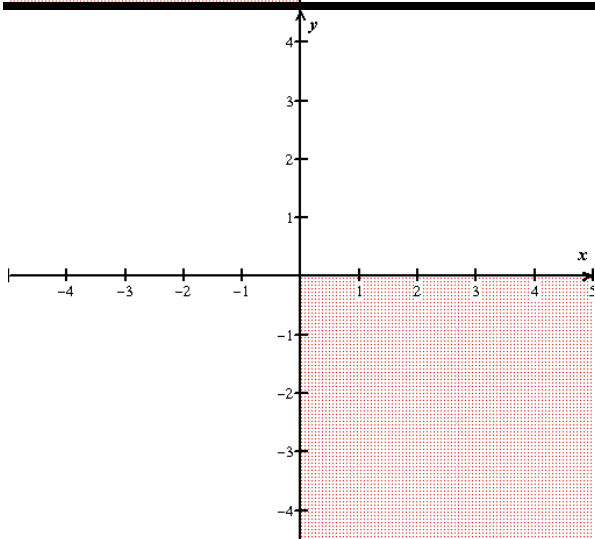
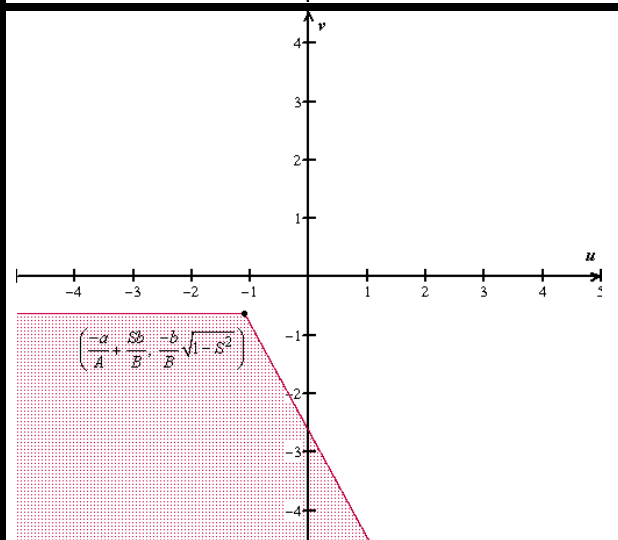


This region in x and y under the equations

$$u = \frac{x-a}{A} - S \left(\frac{y-b}{B} \right)$$

$$v = \sqrt{1-S^2} \left(\frac{y-b}{B} \right)$$

transforms into \Rightarrow
in u and v

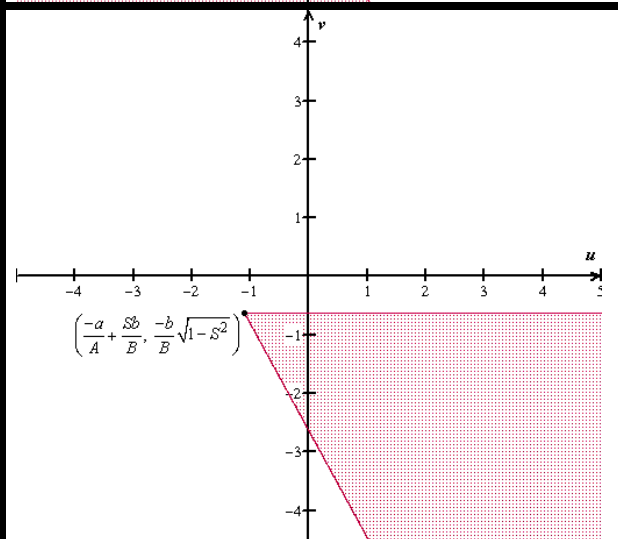


This in region in x and y under the equations

$$u = \frac{x-a}{A} - S \left(\frac{y-b}{B} \right)$$

$$v = \sqrt{1-S^2} \left(\frac{y-b}{B} \right)$$

transforms into \Rightarrow
in u and v



The fourth quadrant boundaries in x, y are $0 < x < \infty; y = 0$ and $x = 0; -\infty < y < 0$. In u, v these become

$$\frac{Sb}{B} - \frac{a}{A} < u < \infty; v = -\frac{b}{B}\sqrt{1-S^2} \text{ and } -\infty < v < -\frac{b}{B}\sqrt{1-S^2}; u = \frac{-a}{A} - \frac{S}{\sqrt{1-S^2}}v.$$

This is illustrated in the table above for $S > 0$. If $S < 0$, the left boundary line in u, v would have a positive rather than a negative slope. If $S = 0$, this boundary becomes a vertical line.

Thus, by an examination of each quadrant the transformation of equation (2) maps the entire $x y$ plane one to one onto the entire $u v$ plane. In fact, such an examination of the quadrant transformations was unnecessary. Since equation (2) is an invertible linear map from R^2 to R^2 it must be both one to one and onto. So the integral of $f(x, y)$ over the $x y$ plane can be evaluated as follows.

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dy dx &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv \\ &= \frac{1}{2\pi AB\sqrt{1-S^2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[-\frac{u^2 + v^2}{2(1-S^2)}\right] \frac{AB}{\sqrt{1-S^2}} du dv \\ &= \frac{1}{2\pi(1-S^2)} \left(\int_{-\infty}^{\infty} \exp\left[-\frac{u^2}{2(1-S^2)}\right] du \right) \left(\int_{-\infty}^{\infty} \exp\left[-\frac{v^2}{2(1-S^2)}\right] dv \right) \\ &= \frac{1}{2\pi(1-S^2)} \left(\int_{-\infty}^{\infty} \exp\left[-\frac{u^2}{2(1-S^2)}\right] du \right)^2 \end{aligned}$$

For positive α , Gaussian and related integrals of this type can be calculated by the following trick of transforming from Cartesian to polar coordinates.

$$\begin{aligned} G(\alpha) &= \int_{-\infty}^{\infty} \exp(-\alpha u^2) du = \sqrt{\left(\int_{-\infty}^{\infty} \exp(-\alpha x^2) dx \right) \left(\int_{-\infty}^{\infty} \exp(-\alpha y^2) dy \right)} \\ &= \sqrt{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp[-\alpha(x^2 + y^2)] dy dx} = \sqrt{\int_0^{2\pi} \int_0^{\infty} \exp(-\alpha r^2) r dr d\theta} \\ &= \sqrt{\left(\int_0^{2\pi} d\theta \right) \left(\int_0^{\infty} \exp(-\alpha r^2) r dr \right)} = \sqrt{(2\pi) \left(\frac{-e^{-\alpha r^2}}{2\alpha} \Big|_0^{\infty} \right)} \\ &= \sqrt{\frac{\pi}{\alpha} [0 - (-1)]} \end{aligned}$$

$$\text{Hence, } G(\alpha) = \int_{-\infty}^{\infty} \exp(-\alpha u^2) du = \sqrt{\pi} \alpha^{-1/2} \quad (5)$$

By symmetry the following odd power integrals must vanish.

$$\int_{-\infty}^{\infty} u^{2n+1} \exp(-\alpha u^2) du = 0 \quad (6)$$

Even power integrals can be calculated from the derivatives of G .

$$\begin{aligned}
 G^{(n)}(\alpha) &= \int_{-\infty}^{\infty} (-u^2)^n \exp(-\alpha u^2) du = (-1)^n \int_{-\infty}^{\infty} u^{2n} \exp(-\alpha u^2) du \\
 \int_{-\infty}^{\infty} u^{2n} \exp(-\alpha u^2) du &= (-1)^n G^{(n)}(\alpha) = (-1)^n \sqrt{\pi} \frac{d^n}{d\alpha^n} (\alpha^{-1/2}) \\
 &= (-1)^n \sqrt{\pi} (-1)^n \frac{1 \cdot 3 \cdots (2n-1)}{2^n} \alpha^{-(2n+1)/2} \\
 &= \sqrt{\pi} \frac{(2n-1)!}{(n-1)! 2^{2n-1}} \alpha^{-(2n+1)/2}
 \end{aligned}$$

$$\text{So, } \int_{-\infty}^{\infty} u^{2n} \exp(-\alpha u^2) du = \sqrt{\frac{\pi}{\alpha^{2n+1}}} \frac{(2n-1)!}{(n-1)! 2^{2n-1}} \quad (7)$$

$$\text{In particular, } \int_{-\infty}^{\infty} u^2 \exp(-\alpha u^2) du = \frac{1}{2} \sqrt{\frac{\pi}{\alpha^3}}.$$

From equation (5) since $\alpha = \frac{1}{2(1-S^2)}$, $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dy dx = \frac{1}{2\pi(1-S^2)} \left(\sqrt{\pi 2(1-S^2)} \right)^2 = 1$. Thus $f(x, y)$ is a valid two variable continuous probability density function.

The marginal probability distributions for x and y are defined as follows: **Note:** here the x subscript on $f_x(x)$ does **not** denote partial differentiation. To avoid this confusion our text refers to this function as $f_1(x)$.

$$f_x(x) = \int_{-\infty}^{\infty} f(x, y) dy \qquad f_y(y) = \int_{-\infty}^{\infty} f(x, y) dx$$

Now, by completing the square,

$$\begin{aligned}
 \frac{\left(\frac{x-a}{A}\right)^2 - 2S\left(\frac{x-a}{A}\right)\left(\frac{y-b}{B}\right) + \left(\frac{y-b}{B}\right)^2}{2(1-S^2)} &= \frac{\left[\left(\frac{y-b}{B}\right) - S\left(\frac{x-a}{A}\right)\right]^2}{2(1-S^2)} + \frac{1}{2}\left(\frac{x-a}{A}\right)^2 \\
 \frac{\left(\frac{x-a}{A}\right)^2 - 2S\left(\frac{x-a}{A}\right)\left(\frac{y-b}{B}\right) + \left(\frac{y-b}{B}\right)^2}{2(1-S^2)} &= \frac{\left[\left(\frac{x-a}{A}\right) - S\left(\frac{y-b}{B}\right)\right]^2}{2(1-S^2)} + \frac{1}{2}\left(\frac{y-b}{B}\right)^2.
 \end{aligned}$$

These identities and a change of variables in the integrals allow the use of equation (5) to simplify the marginal probabilities.

$$\begin{aligned}
 f_x(x) &= \frac{\exp\left[-\frac{1}{2}\left(\frac{x-a}{A}\right)^2\right]}{2\pi AB\sqrt{1-S^2}} \int_{-\infty}^{\infty} \exp\left[-\frac{\left[\left(\frac{y-b}{B}\right) - S\left(\frac{x-a}{A}\right)\right]^2}{2(1-S^2)}\right] dy \\
 &= \frac{\exp\left[-\frac{1}{2}\left(\frac{x-a}{A}\right)^2\right]}{2\pi AB\sqrt{1-S^2}} \int_{-\infty}^{\infty} \exp\left[-u^2 / \left[2(1-S^2)\right]\right] B du = \frac{\exp\left[-\frac{1}{2}\left(\frac{x-a}{A}\right)^2\right]}{2\pi A\sqrt{1-S^2}} \sqrt{2\pi(1-S^2)}
 \end{aligned}$$

$$f_x(x) = \frac{\exp\left[-\frac{1}{2}\left(\frac{x-a}{A}\right)^2\right]}{\sqrt{2\pi A}} \quad (8)$$

$$\begin{aligned} f_y(y) &= \frac{\exp\left[-\frac{1}{2}\left(\frac{y-b}{B}\right)^2\right]}{2\pi AB\sqrt{1-S^2}} \int_{-\infty}^{\infty} \exp\left[-\left[\left(\frac{x-a}{A}\right) - S\left(\frac{y-b}{B}\right)\right]^2 / \left[2(1-S^2)\right]\right] dx \\ &= \frac{\exp\left[-\frac{1}{2}\left(\frac{y-b}{B}\right)^2\right]}{2\pi AB\sqrt{1-S^2}} \int_{-\infty}^{\infty} \exp\left[-u^2 / \left[2(1-S^2)\right]\right] Adu = \frac{\exp\left[-\frac{1}{2}\left(\frac{y-b}{B}\right)^2\right]}{2\pi B\sqrt{1-S^2}} \sqrt{2\pi(1-S^2)} \\ f_y(y) &= \frac{\exp\left[-\frac{1}{2}\left(\frac{y-b}{B}\right)^2\right]}{\sqrt{2\pi B}} \quad (9) \end{aligned}$$

Thus, both marginal probability distributions are Gaussian. Equations (5), (6) and (7) verify that the mean and variance of each marginal distribution are the standard results for a normal distribution.

$\begin{aligned} \mu_x = E(x) = \langle x \rangle &= \int_{-\infty}^{\infty} x f_x(x) dx \\ &= \frac{1}{\sqrt{2\pi A}} \int_{-\infty}^{\infty} x \exp\left[-\frac{1}{2}\left(\frac{x-a}{A}\right)^2\right] dx \\ &= \frac{1}{\sqrt{2\pi A}} \int_{-\infty}^{\infty} (Au + a) \exp\left(-\frac{u^2}{2}\right) Adu \\ &= \frac{1}{\sqrt{2\pi}} (0 + a\sqrt{2\pi}) = a \end{aligned}$	<p>(10)</p>	$\begin{aligned} \mu_y = E(y) = \langle y \rangle &= \int_{-\infty}^{\infty} y f_y(y) dy \\ &= \frac{1}{\sqrt{2\pi B}} \int_{-\infty}^{\infty} y \exp\left[-\frac{1}{2}\left(\frac{y-b}{B}\right)^2\right] dy \\ &= \frac{1}{\sqrt{2\pi B}} \int_{-\infty}^{\infty} (Bu + b) \exp\left(-\frac{u^2}{2}\right) Bdu \\ &= \frac{1}{\sqrt{2\pi}} (0 + b\sqrt{2\pi}) = b \end{aligned}$	<p>(11)</p>
$\begin{aligned} E(x^2) = \langle x^2 \rangle &= \int_{-\infty}^{\infty} x^2 f_x(x) dx \\ &= \frac{1}{\sqrt{2\pi A}} \int_{-\infty}^{\infty} x^2 \exp\left[-\frac{1}{2}\left(\frac{x-a}{A}\right)^2\right] dx \\ &= \frac{A}{\sqrt{2\pi A}} \int_{-\infty}^{\infty} (A^2u^2 + 2Aau + a^2) \exp\left(-\frac{u^2}{2}\right) du \\ &= \frac{1}{\sqrt{2\pi}} (A^2\sqrt{2\pi} + 0 + a^2\sqrt{2\pi}) = A^2 + a^2 \end{aligned}$	<p>(12)</p>	$\begin{aligned} E(y^2) = \langle y^2 \rangle &= \int_{-\infty}^{\infty} y^2 f_y(y) dy \\ &= \frac{1}{\sqrt{2\pi B}} \int_{-\infty}^{\infty} y^2 \exp\left[-\frac{1}{2}\left(\frac{y-b}{B}\right)^2\right] dy \\ &= \frac{B}{\sqrt{2\pi B}} \int_{-\infty}^{\infty} (B^2u^2 + 2Bbu + b^2) \exp\left(-\frac{u^2}{2}\right) du \\ &= \frac{1}{\sqrt{2\pi}} (B^2\sqrt{2\pi} + 0 + b^2\sqrt{2\pi}) = B^2 + b^2 \end{aligned}$	<p>(13)</p>
$\begin{aligned} \sigma_x^2 = E\left[(x - \mu_x)^2\right] &= \langle x^2 - 2\mu_x x + \mu_x^2 \rangle \\ &= \langle x^2 \rangle - 2\mu_x \langle x \rangle + \langle \mu_x^2 \rangle \\ &= \langle x^2 \rangle - 2\mu_x \mu_x + \mu_x^2 \\ &= \langle x^2 \rangle - \mu_x^2 = A^2 + a^2 - a^2 = A^2 \end{aligned}$	<p>(14)</p>	$\begin{aligned} \sigma_y^2 = E\left[(y - \mu_y)^2\right] &= \langle y^2 - 2\mu_y y + \mu_y^2 \rangle \\ &= \langle y^2 \rangle - 2\mu_y \langle y \rangle + \langle \mu_y^2 \rangle \\ &= \langle y^2 \rangle - 2\mu_y \mu_y + \mu_y^2 \\ &= \langle y^2 \rangle - \mu_y^2 = B^2 + b^2 - b^2 = B^2 \end{aligned}$	<p>(15)</p>

The covariance is a measure of the degree of relationship or “lack of independence” between two variables. It is given by

$$\begin{aligned} \text{cov}(x, y) &= E\left(\left[(x - \mu_x)\right]\left[(y - \mu_y)\right]\right) = \langle xy - \mu_x y - \mu_y x + \mu_x \mu_y \rangle \\ &= \langle xy \rangle - \mu_x \langle y \rangle - \mu_y \langle x \rangle + \langle \mu_x \mu_y \rangle \\ &= \langle xy \rangle - \mu_x \mu_y - \mu_x \mu_y + \mu_x \mu_y \\ &= \langle xy \rangle - \mu_x \mu_y \end{aligned}$$

From the transformation equations (2) and (3), the expected value of $\langle xy \rangle$ can be expressed as follows.

$$\begin{aligned} \langle xy \rangle &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f(x, y) dy dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(u, v) y(u, v) F(u, v) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv \\ &= \frac{1}{2\pi AB \sqrt{1-S^2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(a + Au + \frac{SA}{\sqrt{1-S^2}} v \right) \left(b + \frac{B}{\sqrt{1-S^2}} v \right) \exp \left[-\frac{u^2 + v^2}{2(1-S^2)} \right] \frac{AB}{\sqrt{1-S^2}} du dv \\ &= \frac{1}{2\pi(1-S^2)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(ab + \frac{(aB + SAB)v}{\sqrt{1-S^2}} + Abu + \frac{ABuv}{\sqrt{1-S^2}} + \frac{SABv^2}{1-S^2} \right) \exp \left[-\frac{u^2 + v^2}{2(1-S^2)} \right] du dv \end{aligned}$$

Now, from equation (6) the following three integrals vanish.

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u \exp \left[-\frac{u^2 + v^2}{2(1-S^2)} \right] du dv &= \left(\int_{-\infty}^{\infty} \exp \left[-\frac{v^2}{2(1-S^2)} \right] dv \right) \left(\int_{-\infty}^{\infty} u \exp \left[-\frac{u^2}{2(1-S^2)} \right] du \right) \\ &= \sqrt{2\pi(1-S^2)}(0) = 0 \end{aligned}$$

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} v \exp \left[-\frac{u^2 + v^2}{2(1-S^2)} \right] du dv &= \left(\int_{-\infty}^{\infty} \exp \left[-\frac{u^2}{2(1-S^2)} \right] du \right) \left(\int_{-\infty}^{\infty} v \exp \left[-\frac{v^2}{2(1-S^2)} \right] dv \right) \\ &= \sqrt{2\pi(1-S^2)}(0) = 0 \end{aligned}$$

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} uv \exp \left[-\frac{u^2 + v^2}{2(1-S^2)} \right] du dv &= \left(\int_{-\infty}^{\infty} u \exp \left[-\frac{u^2}{2(1-S^2)} \right] du \right) \left(\int_{-\infty}^{\infty} v \exp \left[-\frac{v^2}{2(1-S^2)} \right] dv \right) \\ &= (0)(0) = 0 \end{aligned}$$

Therefore the expected value of $\langle xy \rangle$ simplifies as follows.

$$\begin{aligned} \langle xy \rangle &= \frac{1}{2\pi(1-S^2)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(ab + \frac{SABv^2}{1-S^2} \right) \exp \left[-\frac{u^2 + v^2}{2(1-S^2)} \right] du dv \\ &= \frac{1}{2\pi(1-S^2)} \left[ab \left(\int_{-\infty}^{\infty} \exp \left[-\frac{u^2}{2(1-S^2)} \right] du \right)^2 + \frac{SAB}{1-S^2} \left(\int_{-\infty}^{\infty} \exp \left[-\frac{u^2}{2(1-S^2)} \right] du \right) \left(\int_{-\infty}^{\infty} v^2 \exp \left[-\frac{v^2}{2(1-S^2)} \right] dv \right) \right] \end{aligned}$$

From equation (5), $\int_{-\infty}^{\infty} \exp\left[-\frac{u^2}{2(1-S^2)}\right] du = \sqrt{\pi 2(1-S^2)}$, while from equation (7)

$$\int_{-\infty}^{\infty} v^2 \exp\left[-\frac{v^2}{2(1-S^2)}\right] dv = \frac{\sqrt{\pi 2^3(1-S^2)^3}}{2} = \sqrt{2\pi(1-S^2)}(1-S^2).$$

So the expected value of $\langle xy \rangle$ has the final form.

$$\begin{aligned} \langle xy \rangle &= \frac{1}{2\pi(1-S^2)} \left[ab \left(\sqrt{2\pi(1-S^2)} \right)^2 + \frac{SAB}{1-S^2} \left(\sqrt{2\pi(1-S^2)} \right) \left((1-S^2) \sqrt{2\pi(1-S^2)} \right) \right] \\ &= ab + SAB \end{aligned}$$

Therefore the covariance of x and y is given by

$$\text{cov}(x, y) = \langle xy \rangle - \mu_x \mu_y = ab + SAB - ab = SAB. \quad (16)$$

The correlation coefficient is defined as $\rho = \frac{\text{cov}(x, y)}{\sigma_x \sigma_y}$. It is simply the value of S .

$$\rho = \frac{\text{cov}(x, y)}{\sigma_x \sigma_y} = \frac{SAB}{AB} = S \quad (17)$$

If the variables are independent the covariance and correlation coefficient must be zero. For the bivariate normal distribution a zero covariance ($S = 0$) means that

$$\begin{aligned} f(x, y) &= \frac{1}{2\pi AB} \exp\left[-\frac{\left(\frac{x-a}{A}\right)^2 + \left(\frac{y-b}{B}\right)^2}{2}\right] \\ &= \left(\frac{1}{\sqrt{2\pi A}} \exp\left[-\frac{1}{2}\left(\frac{x-a}{A}\right)^2\right] \right) \left(\frac{1}{\sqrt{2\pi B}} \exp\left[-\frac{1}{2}\left(\frac{y-b}{B}\right)^2\right] \right) \\ &= f_x(x) f_y(y) \end{aligned}$$

which is the precise definition of independent random variables. Thus, for the bivariate normal distribution the random variables are independent **if and only if** the covariance is zero. **Note:** this is not true for an arbitrary joint probability distribution. Independence demands zero covariance and zero correlation. The converse is not always true. It is possible for a distribution of dependent random variables to have zero covariance.