Derivation of the Mean and Standard Deviation of the BinomialDistribution

The purpose of these notes is to derive the following two formulas for the binomial distribution :

(1) $\boldsymbol{\mu = np}$

(2) $\boldsymbol{\sigma = \sqrt{np\,(1-p)}}$.

The starting point for getting (1) is the 'generic' formula true for **any** probability distribution.

(3) $\mu = \sum xP(x)$, where the sum runs over all values of the random variable *x* .

Let the summation index *m* stand for the possible number of successes in *n* identical simple Bernoulli trials in which the probability of a success on any 1 trial is *p*. Thus, we have the random variable $x = m$, and from the binomial distribution,

(4) $$P(m) = b(m; n, p) = \binom{n}{m} p^m (1-p)^{n-m} = \frac{n!}{m!(n-m)!}\, p^m (1-p)^{n-m} \quad .$$

Plugging this probability formula into equation (3), gives the result that

(5) $$\mu = \sum_{m=0}^{n} m\,P(m) = \sum_{m=0}^{n} m \binom{n}{m} p^m (1-p)^{n-m} = \sum_{m=0}^{n} \frac{m\,n!}{m!(n-m)!}\, p^m (1-p)^{n-m} \quad .$$

Since the $m = 0$ term does not contribute to the sum and since $\frac{m}{m!} = \frac{m}{m\,(m-1)!} = \frac{1}{(m-1)!}$, equation (5) can be simplified to

(6) $$\mu = \sum_{m=1}^{n} \frac{n!}{(m-1)!(n-m)!}\, p^m (1-p)^{n-m} = \frac{n!p}{(n-1)!} \sum_{m=1}^{n} \frac{(n-1)!\, p^{m\text{-}1} (1-p)^{n-1-(m-1)}}{(m-1)![(n-1)-(m-1)]!} \quad .$$

Now, let $j = m - 1$, then as *m* runs from 1 to *n* , *j* runs from 0 to $n - 1$. Thus, equation (6) can be written as

(7) $$\mu = \frac{n(n-1)!\,p}{(n-1)!} \sum_{j=0}^{n-1} \frac{(n-1)!\, p^j (1-p)^{n-1-j}}{j![n-1-j]!} = np \sum_{j=0}^{n\text{-}1} \binom{n-1}{j} p^j (1-p)^{n-1-j} \quad .$$

But by the **binomial theorem**, it is true that

(8) $$\sum_{j=0}^{n\text{-}1} \binom{n-1}{j} x^j\, y^{n-1-j} = (x+y)^{n-1}$$. Hence, equation (7) becomes

(9) $$\mu = np\Big(p + (1-p)\Big)^{n-1} = np(\,1\,)^{n-1} = np \;\; .$$

The starting point for getting equation (2) is the corresponding variance formula also true for **any** probability distribution.

(10) $\sigma^2 = \sum (x-\mu)^2\, P(x)$, where again the sum runs over all values of the random variable x .

Squaring the binomial in equation (10) yields the following result:

(11) $$\sigma^2 = \sum (x^2 - 2\mu x + \mu^2)\, P(x) = \sum x^2 P(x) - 2\mu \sum x P(x) + \mu^2 \sum P(x) = \sum x^2 P(x) - 2\mu(\mu) + \mu^2(1) \;\; .$$

Combining like terms in equation (11) finally gives

(12) $\sigma^2 = \sum x^2 P(x) - \mu^2$.

In terms of the summation index m the first sum in equation (12) becomes

$$(13)\quad \sum_{m=0}^{n} m^2\, P(m) \;=\; \sum_{m=0}^{n} m^2 \begin{pmatrix} n \\ m \end{pmatrix} p^m (1-p)^{n-m} \;=\; \sum_{m=0}^{n} \frac{m^2\, n!}{m!(n-m)!}\, p^m (1-p)^{n-m}$$

$$= \sum_{m=0}^{n} \frac{m(m-1+1)\, n!}{m!(n-m)!}\, p^m (1-p)^{n-m}$$

$$= \sum_{m=0}^{n} \frac{m(m-1)\, n!}{m!(n-m)!}\, p^m (1-p)^{n-m} \;+\; \sum_{m=0}^{n} \frac{m(1)\, n!}{m!(n-m)!}\, p^m (1-p)^{n-m} \quad .$$

In the first summation above both the $m = 0$ and the $m = 1$ term vanish, while the second summation is identical to the sum in equation (5). Hence equation (13) simplifies to

$$(14)\quad \sum_{m=0}^{n} m^2\, P(m) \;=\; \sum_{m=2}^{n} \frac{m(m-1)\, n!}{m!(n-m)!}\, p^m (1-p)^{n-m} + \mu$$

$$= \sum_{m=2}^{n} \frac{m(m-1)\, n!}{m(m-1)(m-2)!(n-m)!}\, p^m (1-p)^{n-m} + \mu$$

$$= \sum_{m=2}^{n} \frac{n!}{(m-2)![(n-2)-(m-2)]!}\, p^{m\text{-}2} p^2 (1-p)^{n-2-(m-2)} + \mu \quad , \quad \text{letting } j = m-2$$

$$= p^2 \sum_{j=0}^{n-2} \frac{n!}{j![(n-2)-j]!}\, p^j (1-p)^{n-2-j} + \mu$$

$$= p^2\, n(n-1) \sum_{j=0}^{n-2} \frac{(n-2)!}{j![(n-2)-j]!}\, p^j (1-p)^{n-2-j} + \mu$$

$$= p^2\, n(n-1) \sum_{j=0}^{n-2} \begin{pmatrix} n-2 \\ j \end{pmatrix} p^j (1-p)^{n-2-j} + \mu$$

$$= p^2\, n(n-1) \Big(p + (1-p) \Big)^{n-2} + \mu \;=\; p^2\, n(n-1) \Big(1 \Big)^{n-2} + \mu \quad .$$

Thus, one arrives at the result,

$$(15)\quad \sum_{m=0}^{n} m^2\, P(m) \;=\; p^2\, n(n-1) \;+\; np$$, which when combined with equation (12), yields the following :

$$(16)\quad \sigma^2 \;=\; p^2\,(n^2 - n) \;+\; np \;-\; (np)^2 = p(\, pn^2 \;-\; pn \;+\; n \;-\; n^2 p\,) \;+\; p(n - np) \;=\; np(1-p) \quad .$$

Thus, upon taking square roots one is lead to equation (2).

There is also a 'tricky way' using calculus to derive both equations (1) and (15) $\Big($and hence from equation (12), equation (2) as well$\Big)$. First one defines the following 'generating function' .

$$(17)\quad f(x) = \Big(xp + (1-p) \Big)^n = \sum_{m=0}^{n} \begin{pmatrix} n \\ m \end{pmatrix} (xp)^m (1-p)^{n-m} = \sum_{m=0}^{n} x^m \begin{pmatrix} n \\ m \end{pmatrix} p^m (1-p)^{n-m}$$

Taking the derivative of both sides of equation (17) gives

$$(18) \quad f'(x) = n\Big(xp\ + (1-p)\Big)^{n-1} p = \sum_{m=0}^{n} m\, x^{m-1} \binom{n}{m} p^m (1-p)^{n-m} \quad .$$

By comparison of equation (5) with equation (18), one is lead to equation (2) as follows :

$$(19) \quad \mu = \sum_{m=0}^{n} m\, P(m) = \sum_{m=0}^{n} m \binom{n}{m} p^m (1-p)^{n-m} = f'(1) = n\Big(p\ + (1-p)\Big)^{n-1} p = np\,(1)^{n-1} = np\,.$$

In the same way, to arrive at equation (15), one takes the second derivative of $f(x)$.

$$(20) \quad f''(x) = np(n-1)\Big(xp + (1-p)\Big)^{n-2} p\ = \sum_{m=0}^{n} m(m-1)\, x^{m-2} \binom{n}{m} p^m (1-p)^{n-m}$$

$$(21) \quad f''(x) = np^2(n-1)\Big(xp + (1-p)\Big)^{n-2}$$

$$= \sum_{m=0}^{n} m^2\, x^{m-2} \binom{n}{m} p^m (1-p)^{n-m} - \sum_{m=0}^{n} m\, x^{m-2} \binom{n}{m} p^m (1-p)^{n-m} \quad .$$

Evaluating equation (21) at $x = 1$, and using equation (19) gives the result.

$$(22) \quad f''(1) = np^2(n-1)\Big(p\ + (1-p)\Big)^{n-2} = \sum_{m=0}^{n} m^2 \binom{n}{m} p^m (1-p)^{n-m} - \sum_{m=0}^{n} m \binom{n}{m} p^m (1-p)^{n-m}$$

$$(23) \quad f''(1) = np^2(n-1)\,(1\,)^{n-2} = p^2 n(n-1) = \sum_{m=0}^{n} m^2 \binom{n}{m} p^m (1-p)^{n-m} - np \quad .$$

Hence, one has the identity :

$$(24) \quad p^2 n(n-1) = \sum_{m=0}^{n} m^2\, P(m) - np \quad .$$

Then by solving equation (24) for $\sum_{m=0}^{n} m^2\, P(m)$, one finally is again at equation (15).

$$(25) \quad \sum_{m=0}^{n} m^2\, P(m) = p^2\, n(n-1) + np \quad .$$